Mathematics 172 Homework.

In the last class we looked at a system

$$\Delta S = -bSI$$
$$\Delta I = bSI$$

as a model for the spread of a decease such as head lice where recovering from the infection does not confer immunity. In this model what happens is that S keeps deceasing until all of the population is infected.

We then looked a model where in each time step (which we have been taking to be a day) that some fixed proportion p of the infected population is moves back into the susceptible population. Then the model is

$$\Delta S = -bSI + pI \tag{1}$$
$$\Delta I = bSI - pI \tag{2}$$

Because it is a bit easier to analyze lets look at the continuous version of this

$$\frac{dS}{dt} = -bSI + pI$$
$$\frac{dS}{dt} = bSI - pI$$

where b and p are positive constants and $0 < p < 1$. As usual we start by looking for equilibrium points. That is solve

$$\frac{dS}{dt} = -bSI + pI = I(-bS + p) = 0$$
$$\frac{dS}{dt} = bSI - pI = -I(bS - p) = 0$$

This has the solutions

$$I = 0 \quad \text{or} \quad bS - p = 0.$$

That is all of the points $(S, 0)$ (for any value of S) are equilibrium points as are all of the points $(p/b, I)$ (for any value of I). We could start drawing in arrows to see how points move, but there is an easier way.

Let $N = S + I$ be the total size of the population. This is constant. To double check this we take the derivative:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = (-bSI + pI) + (bSI - pI) = 0.$$

Therefore the derivative of N is zero, which implies N is is constant. So we can solve for I in terms of S and get

$$I = N - S$$

Use this in the equation $\frac{dS}{dt} = I(-bS + p)$ to get

$$\frac{dS}{dt} = (N - S)(-bS + p). \tag{3}$$

This is now a rate equation with just one unknown function and we are experts on these. To make things a bit easier to understand on our first pass through let us look at a case with some numbers. Let

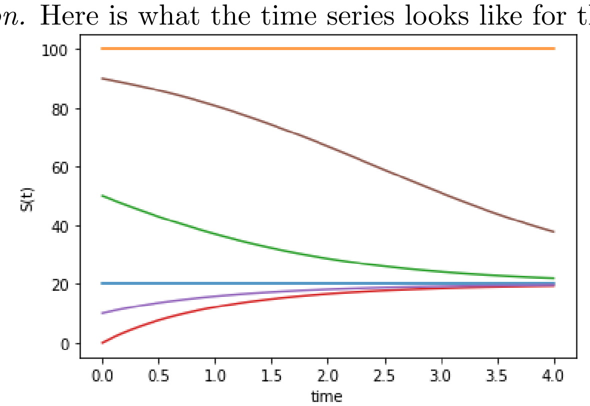
$$b = .01, \qquad p = .2 \qquad N = 100.$$

Then the equation (3) becomes

$$\frac{dS}{dt} = (100 - S)(-.01S + .2).$$

Problem 1. For the last equation show that the equilibrium points are $S = 100$ and $S = 20$. Draw the graphs for the equilibrium solutions and also the solutions with $S(0) = 10$, $S(0) = 50$, and $S(0) = 0$. Use to to show that the point 20 is stable and that the point 100 is unstable. Finally deduce that if $0 < S(0s) < 100$ that $S(t) \approx 20$ for all large t. This in the long run 20% of the population is not infected at any one time. □

Partial solution. Here is what the time series looks like for the solutions:



which shows that the solutions stabilize at $S = 20$. □

Now let us try this with an different set of numbers:

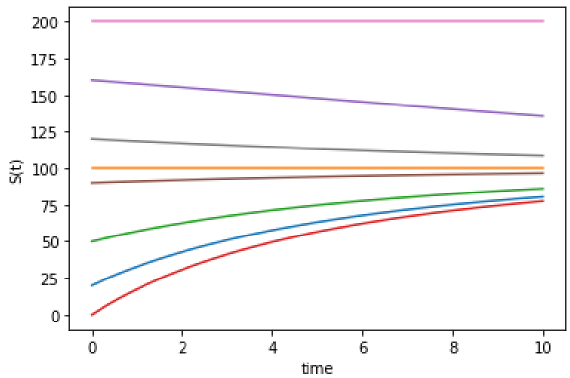
$$b = .001, \qquad p = .2, \qquad N = 100.$$

That is we have changed b to .001 and left the other numbers the same. This time the equation (3) becomes

$$\frac{dS}{dt} = (100 - S)(-.001S + .2).$$

Problem 2. For this new equation show that the two equilibrium points are $S = 100$ and $S = 200$. Draw graphs (the time series) showing that 100 is stable and 200 is unstable. Therefore this time if $0 < S(0) \leq 100$, then $S(t) \approx 100$ for large t. That is in this case the infection dies off. □

Partial solution. Here is what the time series looks like for the solutions:



which shows this time the solutions stabilize at $S = 100$. □

We can now tackle the general case.

Proposition 1. *For the rate equation*

$$\frac{dS}{dt} = (N - S)(-bS + p)$$

with N, b, and p positive constants the equilibrium points are N and $\frac{p}{b}$. The long term behavior splits into two cases:

(a) If $\frac{p}{b} < N$, then $\frac{p}{b}$ is stable and N is unstable. Thus if $0 < S(0) < N$ the long term behavior is that $S(t) \approx \frac{p}{b}$. That is in the long run the number of non-infected individuals in the population stabilizes at $\frac{p}{b}$.

(b) If $N < \frac{p}{b}$, then N is stable and $\frac{p}{b}$ is unstable. Thus if $0 < S(0) \leq N$, then the long term behavior is that $S(t) \approx N$ for large t. That is in the long run the infection dies off.

Problem 3. Draw pictures which explain why this is true. □

Now let us return to the original case of the equations (1) and (2) and as in the continuous case let $N = S + I$. This will be constant. We again solve for I in terms of S to get $I = N - S$. Using this in equation (1) gives

$$\Delta S = (N - S)(-bS + p)$$

which is short hand for

$$S_{t+1} - S_t = (N - S_t)(-bS_t + p)$$

that is

$$S_{t+1} = S_t + (N - S_t)(-bS_t + p) = f(S_t)$$

where

$$f(S) = S + (N - S)(-bS + p).$$

To find the equilibrium points we solve

$$f(S) = S$$

which, using the definition of $f(S)$ and canceling S from both sides, reduces to

$$0 = (N - S)(-bS + p).$$

So in the discrete case we still have that

$$S = N, \qquad S = \frac{p}{b}$$

are the equilibrium points.

As a bit of review, recall that when for a system $S_{t+1} = f(S_t)$, if $S = S_*$ is an equilibrium point, then S_* is stable if $|f'(S_*)| < 1$ (that is $-1 < f'(S_*) < 1$) and it is unstable if $|f'(S_*)| > 1$ (that is $f'(S_*) < -1$ or $f'(S_*) > 1$). So we wish to compute the derivative of $f(S) = S + (N - S)(-bS + p)$.

Problem 4. Use the product rule to show

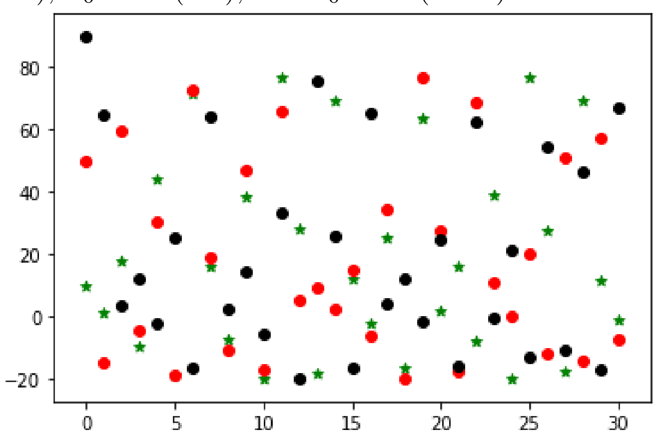
$$f'(S) = 1 - (-bS + p) - b(N - S)$$

and therefore

$$f'(N) = 1 + (bN - p)$$
$$f'\left(\frac{p}{b}\right) = 1 - (bN - p)$$

□

Case 1: $|bN - p| > 1$. This is the crazy case where neither of the equilibrium points are stable. Thus solutions are going to jump around almost at random.[1] As an example here is a plot with $N = 100$, $b = .03$, and $p = .2$, so that $bN - p = 2.8 > 1$ using the three initial conditions $S_0 = 10$ (green), $S_0 = 50$ (red), and $S_0 = 90$ (black) for 30 time steps.

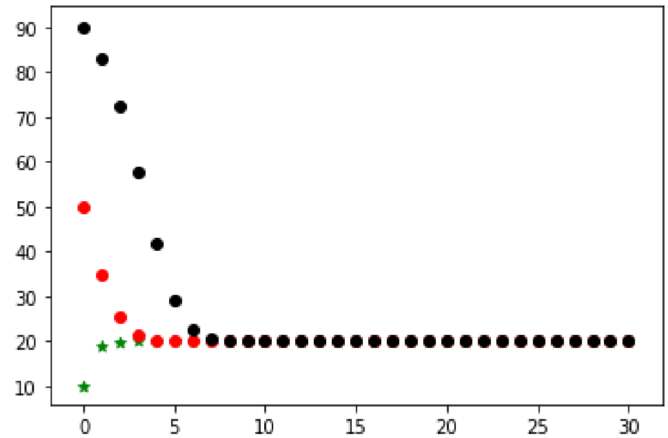


As you can see there is no obvious pattern, and the model also predicts negative vales of S, which makes no sense biologically. □

Case 2: $bN - p > 0$ and $bN - p < 2$. Then $f'(N) = 1 + (bN - p) > 0$, so the equilibrium point N is unstable, and $f'(p/b) = 1 - (bN - p)$ has $|f'(p/b)| < 1$ so p/b is stable. Therefore in this case the long term behavior is that the solutions stabilize at $S = p/b$. Here is the time series for the case $N = 100$, $b = .01$ and $p = .2$ where $bN - p = .8$ with the same initial

[1]This is not quite true, there will be some vales of N, b, and p where there are stable period orbits but discussing this would take us too far afield.

conditions $S_0 = 10$ (green), $S_0 = 50$ (red), and $S_0 = 90$ (black) for 30 time steps where it is clear that the solutions stabilize at $p/b = 20$.



Case 3: $bN - p < 0$ and $-2 < bN - p$. Then $f'(N) = 1 - (bN - p)$ has $-1 < f'(N) < 1$ so N is stable. And $f'(p/b) = 1 - (bN - p) > 1$ so p/b is unstable. This time the long term behavior is that solution stabilize at $S = N$, that is the infection dies off. Here is the time series for $N = 100$, $b = .001$, and $p = .2$ in which case $bN - p = -.1$ with the same initial conditions and color scheme as before.