# Mathematics 172 Homework, August 29, 2019.

A quick review of what we have done to date is that if we have a rate equation (also called a differential equation) that satisfies

$$\frac{dy}{dy} = ry$$

where $r$ is a constant, then

$$y(t) = y(0)e^{rt}$$

so that if $r > 0$ we have exponential growth and if $r < 0$ we have exponential decay. The constant $r$ can be interpreted as follows. The derivative

$$\frac{dy}{dt}$$

is the rate of change of the total population size, $y$, with respect to $t$. Then

$$r = \frac{1}{y}\frac{dy}{dt}$$

is the ***relative growth rate*** or ***intrinsic growth rate***. That is total rate of change divided by the total population size. Very roughly it can be thought of as the rate of change per unit of the population.

Using

$$\frac{dP}{dt} = rP$$

for a model of continuous population growth with no constraints on the population growth is reasonable for small population sizes for a short period of time. Thus if algae is introduced to a pond with lots of nutrients for it and no predators, then we can expect unconstrained population growth for maybe the first several weeks, but eventually the population will become large enough that unconstrained growth is not reasonable.

One way to deal with this problem is to realize that the relative growth rate

$$r = \frac{1}{y}\frac{dy}{dt}$$

will depend on the population size. So be make the assumption that

$$r = \rho(y)$$

depends on the size of the population. Clearing of fractions this leads to a differential equation of the form

$$\frac{dy}{dt} = y\rho(y).$$

So before doing the biological modeling let us get use to working with rate equations of the form

$$\frac{dy}{dt} = f(y).$$

In analysing these equation we will use some basic facts about the derivative:

- If $y' > 0$, then the function is increasing.
- If $y' < 0$, then the function is decreasing.
- The function $y$ is constant if and only if $y' \equiv 0$.

As a first example consider

$$\frac{dy}{dt} = .1y(10 - y). \tag{1}$$

**1.** (a) If $y(0) = 3$ find the derivative $y'(0)$. *Solution:* From the equation we have

$$y'(0) = .1y(0)(10 - y(0)) = .1(3)(10 - 3) = 2.1$$

(b) If $y(5.2) = 15$ find $y'(5.2)$. *Solution:* Again use the equation

$$y'(5.2) = .1y(5.2)(10 - y(5.2)) = .1(15)(10 - 15) = -7.5$$

(c) If $y(4) = 10$, find $y'(4)$. *Solution:* And the pattern is the same, use the equation:

$$y'(4) = .1y(4)(10 - y(4)) = .1(10)(10 - 10) = 0.$$

(d) Show that the constant functions $y = 0$ and $y = 10$ are solutions to the equation (1). *Solution:* We do this for $y = 10$. This is a constant so $y' = 0$. But also $.1(10)(10 - 10) = 0$. Therefore both sides of (1) are equal to 0 for this function. $\square$

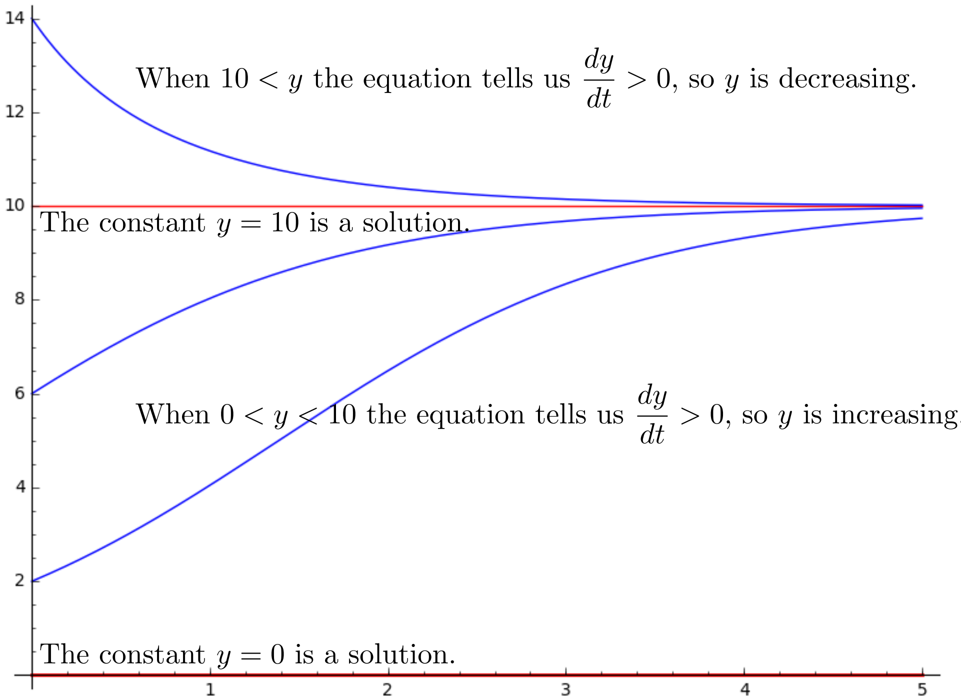
We now do a bit more analysis of (1). If $0 < y < 10$, then

$$y' = \frac{dy}{dt} = .1y(10 - y)$$

is positive and therefore $y$ is increasing. If $10 < y$, then

$$y' = \frac{dy}{dt} = .1y(10 - y) < 0$$

which implies that $y$ is decreasing. We can now get a pretty good idea of what solution to the equation look like:


When $10 < y$ the equation tells us $\frac{dy}{dt} > 0$, so $y$ is decreasing.
The constant $y = 10$ is a solution.
When $0 < y < 10$ the equation tells us $\frac{dy}{dt} > 0$, so $y$ is increasing.
The constant $y = 0$ is a solution.
14
12
10
8
6
4
2
1
2
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5