Mathematics 172 Homework.

We have become experts on analysing differential equations of the form

$$\frac{dP}{dt} = f(P)$$

by finding the equilibrium points, using where $f(N)$ is positive and negative to see where the solutions are increasing or decreasing, and using this information to see which of the equations points are stable. Here we will use these ideals to set up and analyse some problems related to population growth.

The solutions to the problems appear at the end of this document. You should try the problems before looking at the solutions.

Problem 1. Algae is being grown in a tank to be dried and used as fish food. The algae is being harvested so that it is its intrinsic growth rate is $r = -.08$ (grams/day)/gram. Let $W(t)$ be the weight of the algae in the tank at after t days.

(a) What is the rate equation satisfied by $W(t)$.
(b) To keep the population of algae from dying out the tank is stocked at a constant rate of 400 grams/day. What is the new rate equation for W?
(c) What are the equilibrium point(s) of the new equation?
(d) Which of the equilibrium point(s) are stable?
(e) What is the long term fate of the algae population? □

Problem 2. With the same set up as the last problem, that is algae in a tank with a relative growth rate of $r = -.08$, assume that the owner of the tank wishes to have a stable population of 2,000 grams of algae in the tank. At what rate should she stock the tank? □

Recall that the logistic equation with intrinsic growth rate r and carrying capacity K is

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

Problem 3. Here is a variant on the problem we started last class. Assume that a population of Paramecium is growing a small puddle. Let $P(t)$ be the weight of the population of Paramecium in the puddle after t days. Assume this population the population grows logistically with an intrinsic growth rate of .4 (grams/day)/gram with a carrying capacity of 3 grams.

(a) What is the differential equation satisfied by P?
(b) At some point some rotifers invade the puddle. Assume that they eat the Paramecium at the rate of .2 grams/day. What is the new rate equation?
(c) What happens to the population of Paramecium? □

Problem 4. With the same starting set up as the last problem assume that the rotifers eat 15% Paramecium population per day rather than just a fixed amount. What happens this time? □

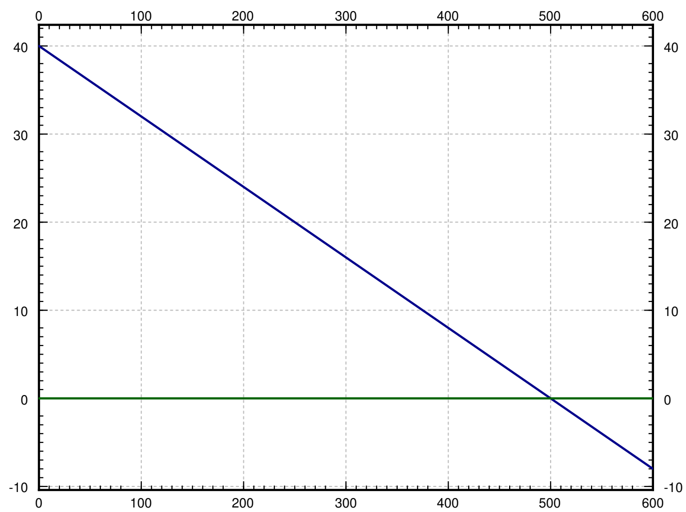
Solution to Problem 1. (a) The rate equation for W is

$$\frac{dW}{dt} = -.08W$$

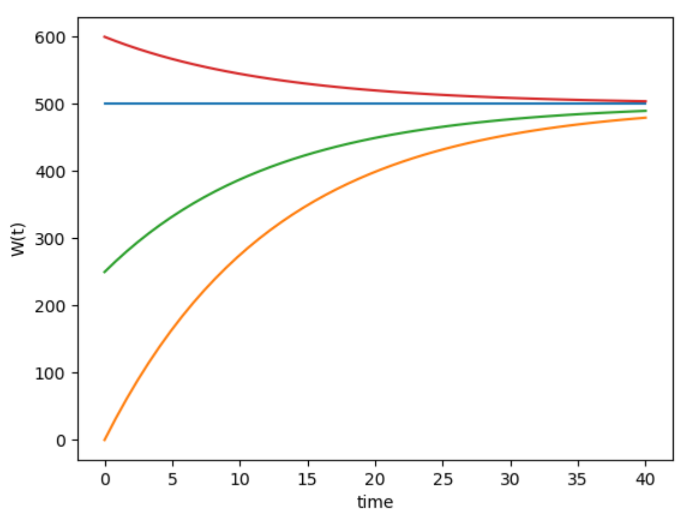
(b) The new rate equation is

$$\frac{dW}{dt} = -.08W + 40.$$

(c) Solving $-.08W + 40 = 0$ gives that there is only one equilibrium point and it is $W = 40/.08 = 500$.
(d) Plotting dW/dt as a function of N gives the graph:



From this we see that $dW/dt > 0$ for $W < 500$ and $dW/dt < 0$ for $W > 500$. So the graph of solutions (the time series) looks like:



Therefore we see that $W = 500$ is stable and that all solution converge to $W = 500$ as t gets large.

(e) In the long run the population of algae stabilizes at a size of 500 grams. □

Solution to Problem 2. As before the rate equation satisfied by W before stocking is

$$\frac{dW}{dt} = -.08W.$$

Let S grams/day be the stocking rate. Then the new rate equation is

$$\frac{dW}{dt} = -.08W + S.$$

We wish $W = 2{,}000$ to be a stable equilibrium point. Then

$$-.08(2{,}000) + S = 0.$$

This gives

$$S = -.08(2{,}000) = 160\text{grams/day}$$

as the required stocking rate. □

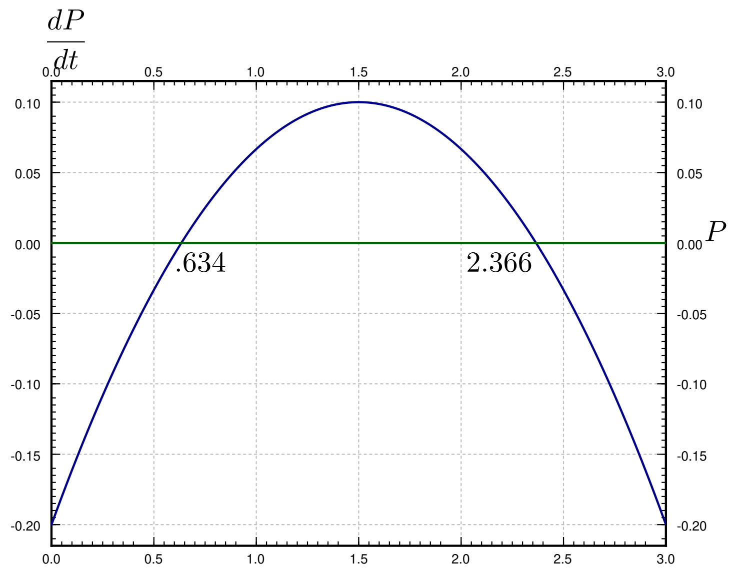
Solution to Problem 3. (a) The logistic equation for P is

$$\frac{dP}{dt} = .4P\left(1 - \frac{P}{3}\right).$$

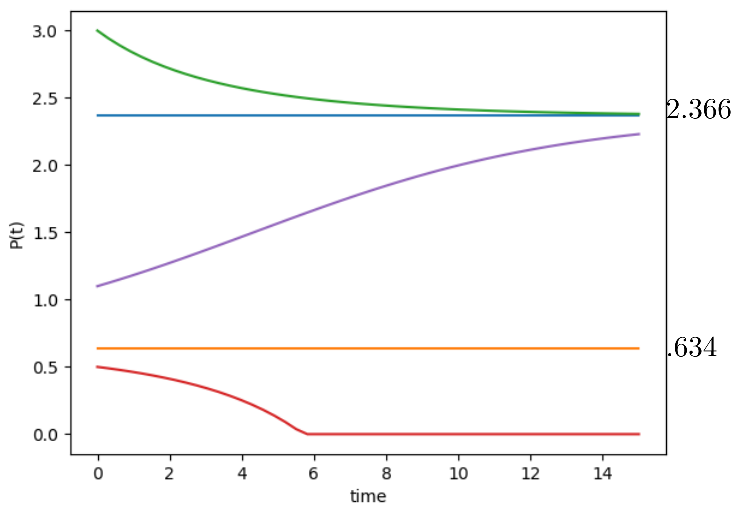
(b) The new rate equation is

$$\frac{dP}{dt} = .4P\left(1 - \frac{P}{3}\right) - .2$$

(c) To understand what happens we look for the equilibrium points. To do this we plot dP/dt as a function of P for $0 \leq P \leq 3$ on the calculator and find use the graph to find the equilibrium points:



So the equilibrium points are $P = .634$ and $P = 2.366$. The time series (graph of P as a function of t) then looks like:



Thus $P = 2.366$ is stable and $P = .634$ is unstable. Thus, assuming that the population of Paramecium was close to the carrying capacity of $K = 3$ grams when the rotifers arrived, the population stabilized at 2.366 grams after the rotifers invaded. $\square$

Solution to Problem 4. This time the new rate equation is

$$\frac{dP}{dt} = .4P\left(1 - \frac{P}{3}\right) - .15P.$$

Solving

$$.4P\left(1-\frac{P}{3}\right)-.15P=0$$

yields that the equilibrium points are $P=0$ and $P=15/8=1.875$. You can check that $P=1.875$ is stable. Thus this time the population stabilizes at 1.875 grams.