Mathematics 172 Homework.

Our goal for the immediate future is to lean how to use Euler's method to approximate the solutions to differential equations. To start we recall some facts about the derivative. Let $y = f(x)$. Then the derivative of f at $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(x+h) - f(a)}{h}$$

which is just a fancy way of saying that for small h we have the approximation

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

We can solve this for $f(a+h)$ to get

$$f(a+h) \approx f(a) + f'(a)h. \tag{1}$$

Problem 1. If $f(1) = 2$ and $f'(1) = .5$ estimate $f(1.1)$, $f(1.05)$ and $f(.95)$.

Solution. We have $f(1.1) = f(1 + .01)$ so let $a = 1$ and $h = .1$ in our basic approximation equation (1) to get

$$f(1.1) \approx f(1) + f'(1)(.1) = 2 + .5(.1) = 2.05$$

Likewise

$$f(1.05) \approx f(1) + f'(1)(.05) = 2 + .5(.05) = 2.025$$

and

$$f(.95) = f(1 + (-.05)) \approx f(1) + f'(1)(-.05) = 2 + .5(0.05) = 1.975$$

□

Now consider a differential equation

$$y' = f(y).$$

Then if we know $y(a)$, we can compute $y'(a)$ from the equation:

$$y'(a) = f(y(a)).$$

Thus for small h we have

$$y(a+h) \approx y(a) + y'(a)h = y(a) + f(y(a))h. \tag{2}$$

Let us consider a simple equation

$$y' = -.1y + 2.$$

Assume that $y(1) = 3$ and estimate $y(1.1)$. From Equation (2) with $a = 1$ and $h = .1$ we have

$$y(1.1) \approx y(1) + (-.1y(1) + 2)h = 3 + (-.1(3) + 2)(.1) = 3.17$$

Problem 2. For the differential equation

$$y' = 2y - 3$$

if $y(2) = 5$ estimate $y(2.1)$ and $y(2.01)$.

Solution. For $y(2.1)$ we have $a = 2$ and $h = .1$ and thus

$$y(2.1) \approx y(2) + (2y(2) - 3)(.1) = 5 + (2(5) - 3)(.1) = 5.7$$

For $y(2.01)$ we have $h = .01$ and so this time the approximation becomes

$$y(2.01) = 5 + (2(5) - 3)(.01) = 5.07$$

□

Back to the general case of an initial value problem:

$$y' = f(y), \qquad y(t_0) = y_0 \tag{3}$$

where t_0 and let h be a small positive number. Let t_1 be

$$t_1 = t_0 + h.$$

By what we have just done have the approximation

$$y(t_1) = y(t_0 + h) \approx y(t_0) + f(y(t_0))h.$$

Now set

$$y_1 = y(t_0) + f(y(t_0))h.$$

This should be a good approximation to the $y(t_1)$. Now set

$$t_2 = t_1 + h = t_0 + 2h.$$

Then the same reasoning gives that

$$y_2 = y_1 + f(y_1)h$$

is a good approximation to $y(t_2)$. We can keep going like this. In general if t_k and y_k have been defined then set

$$\begin{aligned} t_{k+1} &= t_k + h \\ y_{k+1} &= y_k + f(y_k)h. \end{aligned}$$

Then y_n should be a good approximation to $y(t_n)$ where y is the solution to the initial value problem (3). This is ***Euler's Method***.

Problem 3. Do three steps of length $h = .1$ of Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 2.$$

Solution. Here $t_0 = 0$, $y_0 = 2$, and $f(y) = 1 + y$. The general step in Euler's method becomes

$$\begin{aligned} t_{k+1} &= t_k + .1 \\ y_{k+1} &= y_k + (1 + y_k)(.1). \end{aligned}$$

Therefore

$$\begin{aligned} t_1 &= t_0 + h = 0 + .1 = .1 \\ y_1 &= y_0 + (1 + y_0)h = 2 + (1 + 2)(.1) = 2.3 \\ t_2 &= t_1 + h = .1 + .1 = .2 \\ y_2 &= y_1 + (1 + y_1)h = 2.3 + (1 + 2.3)(.1) = 2.63 \\ t_3 &= t_2 + h = .2 + .1 = .3 \\ y_3 &= y_2 + (1 + y_2)h = 2.63 + (1 + 2.63)(.1) = 2.993 \end{aligned}$$

□

We could keep this up for 10 steps (which I have done on a computer) to get that

$$t_{10} = 1.0 \qquad y_{10} = 6.78122$$

which gives us the approximation

$$y(1) \approx 6.78122$$

for the solution to the initial value problem.

This is the theory behind Euler's method. We will shortly see how to make the calculator take as many steps as we need.