Mathematics 172 Homework.

Back to Euler's method for the initial value problem:

$$y' = f(y), \qquad y(0) = y_0.$$

Let us recall the basic theory. If h is a small number then we have the tangent line approximation

$$y(t+h) \approx y(t) + y'(t)h. \tag{1}$$

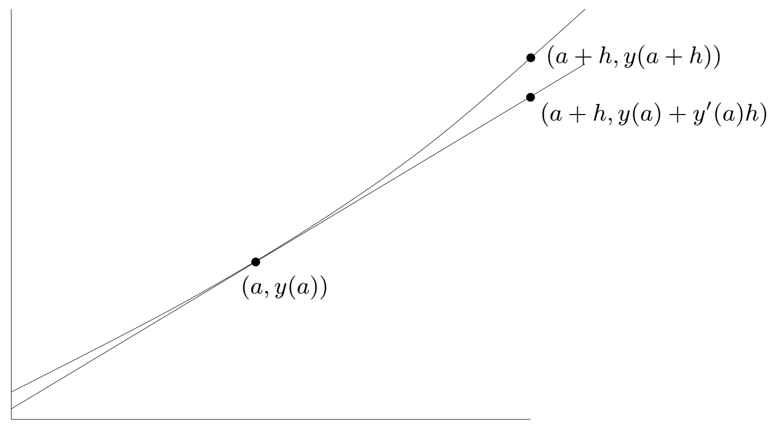


FIGURE 1. This shows the graph of y as a function of t along with the tangent line to this graph at the point $(a, y(a))$. When $t = a + h$ the point $(a+h, y(a+h))$ is on the graph point $(a+h, y(a)+y'(a)h)$ is on the tangent line. When h is small these two points are very close together. Translating this geometric fact into formulas gives equation (1). Note as h gets smaller these points get even closer together.

But we are assuming that y satisfies the differential equation $y' = f(y)$ so in this case the tangent line approximation can be rewritten as

$$y(t+h) \approx y(t) + f(y(t))h. \tag{2}$$

Let us call this ***a basic Euler approximation of length*** h for the differential equation $y' = f(y)$.

Let $h > 0$ be a small positive number which we will refer to as the ***step size***. Set $t_0 = 0$. Now define

$$\begin{aligned} t_1 &= t_0 + h \\ y_1 &= y_0 + f(y_0)h. \end{aligned}$$

This is our first step in Euler's method. By the basic Euler approximation we have

$$\begin{aligned} t_1 &= t_0 + h \\ y_1 &\approx y(t_1). \end{aligned}$$

The second step is to set

$$t_2 = t_1 + h$$
$$y_2 = y_1 + f(y_1)h.$$

Since this is anther basic Euler approximation of length h:

$$t_2 = t_1 + h = t_0 + 2h$$
$$y_2 \approx y(t_2).$$

The third step is

$$t_3 = t_2 + h = t_0$$
$$y_3 = y_2 + f(y_2)h$$

leading to the

$$t_3 = t_0 + 3h$$
$$y_3 \approx y(t_3).$$

and (as you have no doubt already figured out) the fourth step is

$$t_4 = t_3 + h$$
$$y_4 = y_3 + f(y_3)h.$$

giving the approximation

$$t_4 = t_0 + 4h$$
$$y_4 \approx y(t_4).$$

In general once we have taken $n-1$ steps (so that we have computed t_{n-1} and y_{n-1}) the next step is

$$t_n = t_{n-1} + h$$
$$y_n = y_{n-1} + f(y_{n-1})h.$$

It is not hard to get a formula for t_n: taking n steps of size n covers a distance of nh and thus

$$t_k = t_0 + nh$$

Now let use get the calculator to do this. We now need to set up the calculator to work with tables. To do a concrete example let us use the simple equation

$$y' = -.5y + 3 \qquad y(0) = 2.$$

To start press the MODE key. This should open up a screen that looks something like this (some of the highlighted boxes may be in different places):

NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi re^θi
FULL HORIZ G-T

Use the cursor key to move down to the forth line and over to SEQ and press enter to change from FUNC mode to SEQ mode. The screen will now look like:

NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi re^θi
FULL HORIZ G-T

Now press 2ND TABLESET and edit until it looks like

TABLE SETUP
TblStart=0
ΔTbl=1
Indpnt : Auto Ask
Depend: Auto Ask

For this example we will use a step size of

$$h = .1$$

Let us store this the in H register. To do this go to the main screen and enter .1 then push STO followed by ALPHA and H.

We next enter the equation. Press the Y= bottom. The screen will now look something like

Plot1 Plot2 Plot2
nMin=
\u(n)=
u(nMin)=
\v(n)=
v(nMin)=
\w(n)=
w(nMin)=

In calculator notation we use u for the dependent variable y. For the equation we are using the basic Euler step is

$$y_n = y_{n-1} + (-.5y_{n-1} + 3)h.$$

In entering this into the calculator here are some points to keep in mind:

- Where there is an n use the X,T, θ, n key.
- For u, v, and w use 2ND u (over the 7 key), 2ND v (over the 8 key), and 2ND w (over the 9 key).
- For H use press ALPHA H. (When we run the program the calculator is smart enough to use our stored value h.)

```
 Plot1 Plot2 Plot2
 nMin=0
\u(n)=u(n-1) + (-.5 u(n-1) + 3) H
 u(nMin)=2
\v(n)=
  v(nMin)=
\w(n)=
  w(nMin)=
```

And we are now almost done. Press 2ND TABLE and you get output that looks like

n	$u(n)$
0	2
1	2.2
2	2.39
3	2.5705
4	2.742
5	3.9049
6	3.0596

You can now scroll down and find that

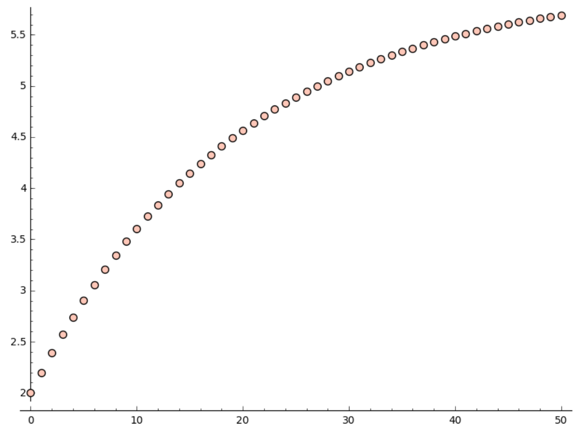
$$u(10) = 3.6051$$

In terms of our original problem this tells use that if we take 10 steps of size $h = .1$ we get the approximation

$$y(1) \approx 3.6051$$

to the solution to the initial value problem $y' = -.5y + 3$, with $y(0) = 2$.

We can also graph the solution. Use the WINDOW button and set nMin=0 and nMax =50. As the step size is $h = .1$ the n value of $n = 50$ corresponds to $t = 5$. Now do a ZOOM 0:ZoomFit. You should then get a graph that looks something like:



You can get $u(10) \approx y(1)$ off of the graph by doing 2ND CALC 1:value and giving the calculator the value $n = 10$. The result is at the bottom of the screen where you have X=10 (which is just the n value) and Y = 3.6050522 which is $u(10) \approx y(1)$.

Problem 1. To get a more accurate approximation to this initial value problem we could take 20 Euler steps of size $h = .05$. Do this. Then approximate $y(.45)$ by taking 45 Euler steps of size $h = .01$.

Solution. The only we need to change is to store .05 in the H register. You then get $u(20) = 3.5892$ which will be a better approximation that our original 3.0596.

To do the approximation of $y(.45)$ store .01 in the H and then find that $u(45) = 2.8077$ which will be pretty close to the exact solution to the problem. □

Remark 1. The exact solution to $y' = -.5y + 3$ is

$$y = 6 - 4e^{-.5t}.$$

This gives $y(.45) = 2.80593512496$ so the estimate of $u(45) = 2.8077$ is accurate to almost 4 sufficient digits. The relative error is $(u(45)-y(.45))/y(.45) = .0006289$. So the approximation is good enough for almost any applicator I can think of. □

Let us return to the logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

Let us consider the case where

$$r = .15, \qquad K = 100, \qquad P(0) = 90.$$

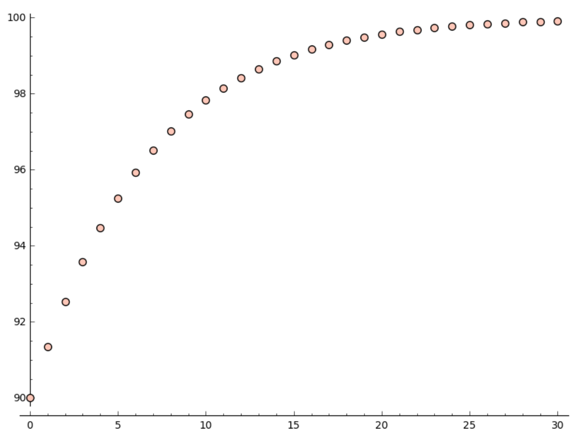
Store .015 in the R register (that is enter .015 on the main screen, then push STO then ALPHA R. Likewise store 100 in the K register. And let us use a step size of $h = .1$, so store .1 in the H register. Enter the equation and initial

Plot1 Plot2 Plot2
nMin=0
\u(n)=u(n-1) + Ru(n-1)(1 - u(n-1)/K)H
u(nMin)=90

condition as

Problem 2. With this set up estimate $P(2.5)$.

Solution. Since we are using a step size of $h = .1$ to get to 2.5 we need 25 steps. We could now use table to find $u(25)$. But I am going to use the graph. Set nMin=0 and nMax=30. Do a ZOOM, 0:ZoomFit to get a graph that looks like



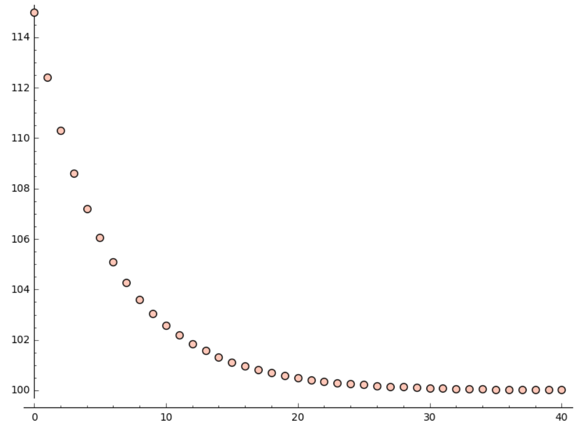
Now 2ND CALC 1:value $n = 25$ gives

$$u(25) = 99.805789$$

as the approximate of $P(2.5)$. □

Problem 3. Using the same equation and still using $h = .1$, but with the initial condition $P(0) = 115$, graph the solution with $0 \le t \le 4$ and estimate $P(.5)$, $P(2.5)$ and $P(4)$.

Solution. The only changes that need to be made are setting \u(nMin)=115 and nMax=40. The graph will look like



and we can use 2ND CALC 1:value to get the approximations

$$P(.5) \approx u(5) = 106.04955$$
$$P(2.5) \approx u(25) = 100.21938$$
$$P(4) \approx u(40) = 100.01912$$

□