Mathematics 172 Homework.

In class today we looked at the discrete version of unconstrained population growth. Let N_t be the population size in year t. Then we looked at the equation

$$N_{t+1} = \lambda N_t \tag{1}$$

where λ is a constant. One way to motivate this is to let r be the ***per capita growth rate***. For example if $r = .15$ (which corresponds to a 15% growth rate) and we have a population of size N_t in year t, then the population the next year is

$$N_{t+1} = N_t + .15N_t = (1.15)N_t.$$

Therefore if we know the population in some year, then to find the population then next year we just have to multiply by 1.15.

In general if the per capita growth rate is r, then

$$N_{t+1} = N_t + rN_t = (1 + r)N_t.$$

We then use the abbreviation

$$\lambda = 1 + r$$

and call λ the ***growth ratio***. If N_t solves (1) then knowing the population size, N_t, in year t let us compute the population size the next year, N_{t+1}, by just multiplying by the growth ratio λ. Therefore we use this starting with N_0 to get

$$\begin{aligned}
N_1 &= \lambda N_0 = N_0\lambda \\
N_2 &= \lambda N_1 = \lambda N_0\lambda = N_0\lambda^2 \\
N_3 &= \lambda N_2 = \lambda N_0\lambda^2 = N_0\lambda^3 \\
N_4 &= \lambda N_3 = \lambda N_0\lambda^3 = N_0\lambda^4 \\
N_5 &= \lambda N_4 = \lambda N_0\lambda^4 = N_0\lambda^5 \\
N_6 &= \lambda N_5 = \lambda N_0\lambda^5 = N_0\lambda^6 \\
N_7 &= \lambda N_6 = \lambda N_0\lambda^6 = N_0\lambda^7
\end{aligned}$$

and at this point you see the pattern:

$$N_t = N_0\lambda^t. \tag{2}$$

Problem 1. A population of 20 killifish is introduced into a pond. These fish breed just one a year and only live a year. Assume that the per capita grow of the fish is .31 fish/fish.

(a) Give a formula for the number of fish after t years. *Solution:* If N_t is the population size, then the growth ratio is $\lambda = 1 + .31 = 1.31$ and $N_0 = 20$. Therefore $N_t = N_0\lambda^t = 20(1.31)^t$.

(b) How long until there are 2,000 fish? *Solution:* We need to solve $N_t = 20(1.31)^t = 2{,}000$. The solution is

$$t = \frac{\ln(2{,}000/20)}{\ln(1.31)} = 17.05\text{years}.$$

So to guarantee 2,000 should wait to year 18. $\square$

Proposition 1. *In a different pond* 50 *of the killifish are released. The population is counted five years later and there are* 213 *of the fish. What is the per capita growth rate of the population?* Solution: *If N_t is the population size and λ is the growth ratio, then $N_t = 50\lambda^t$. To find λ set*

$$N_5 = 50\lambda^5 = 231.$$

This gives

$$\lambda = \left(\frac{231}{50}\right)^{1/5} = 1.3581$$

and therefore the per capita is $r = \lambda - 1 = .3581$. $\square$

We are also going to want to look at more general equations of the form

$$N_{t+1} = f(N_t).$$

That is where if we know the population, N_t, in year t, then there is a function, f, so that we can find the population size in the next year by applying f to N_t.

Here is an example. Assume that

$$N_{t+1} = N_t + .5N_t\left(1 - \frac{N_t}{100}\right) \qquad N_0 = 50.$$

Then

$$N_1 = N_0 + .5N_0\left(1 - \frac{N_0}{100}\right) = 50 + .5(50)\left(1 - \frac{50}{100}\right) = 62.5$$
$$N_2 = N_1 + .5N_1\left(1 - \frac{N_1}{100}\right) = 62.5 + .5(62.5)\left(1 - \frac{62.5}{100}\right) = 74.219$$
$$N_3 = N_2 + .5N_2\left(1 - \frac{N_2}{100}\right) = 74.219 + .5(74.219)\left(1 - \frac{74.219}{100}\right) = 83.78$$
$$N_4 = N_3 + .5N_3\left(1 - \frac{N_3}{100}\right) = 83.7860g + .5(83.7860)\left(1 - \frac{83.7860}{100}\right) = 90.57854$$

and we can go on indefinitely.

Problem 2. If

$$P_{t+1} = \frac{4P_t}{1 + .1(P_t)^2} \qquad P_0 = 3$$

find P_1, P_2 and P_3. *Solution:* $P_1 = 6.3158$, $P_2 = 5.0639$, and $P_3 = .6829$. $\square$