Mathematics 172

Quiz 10 **Name: Solution key**

1. Potassium-argon dating, abbreviated KAr dating, is a radiometric dating method based on the fact that the radioactive decay of an isotope of potassium (K), into to isotope ^{40}Ar of argon has a half life of 1.248 billion years. A sample of mica is found to have 25% of its original potassium left. How old is the mica sample?

Solution: Let P_0 be the initial amount of potassium. Then

$$P(t) = P_0 e^{rt}$$

where r is the intrinsic growth rate and t is measured in billions of years. Then

$$P(1.248) = P_0 e^{1.248r} = \frac{1}{2} P_0 = .5P_0.$$

Canceling the P_0 leads to

$$e^{1.248r} = .5$$

Taking the natural logarithm gives

$$1.248r = \ln(.5)$$

and therefore

$$r = \frac{\ln(.5)}{1.248} = -.5554$$

and therefore

$$P(t) = P_0 e^{-.5554t}.$$

To find the age of our sample we need to solve

$$P(t) = P_0 e^{-.5554t} = .25P_0$$

The P_0's cancel which leaves

$$e^{-.5554t} = .25$$

Taking ln and dividing by $-.5554$ gives

$$t = \frac{\ln(.25)}{-.5554} = 2.496 \text{ billion years}$$

as the age of the sample. □

2. For the rate equation

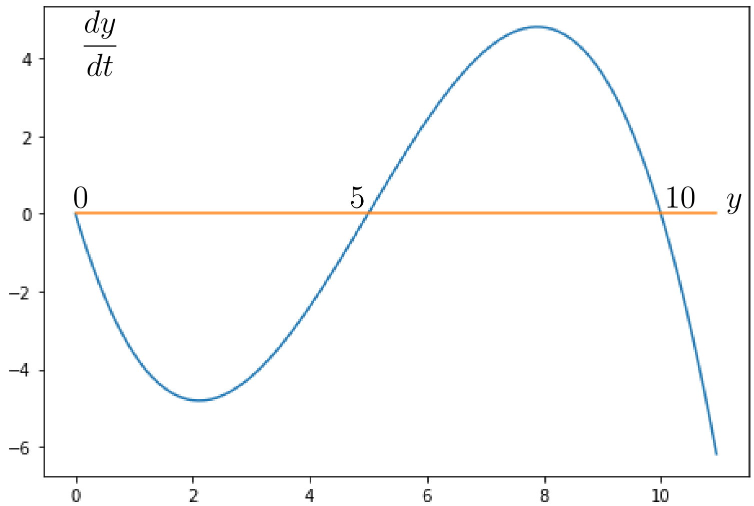
$$\frac{dy}{dt} = .1y(y-5)(10-y)$$

(a) Find the equilibrium solutions and classify them as to stable and unstable points.

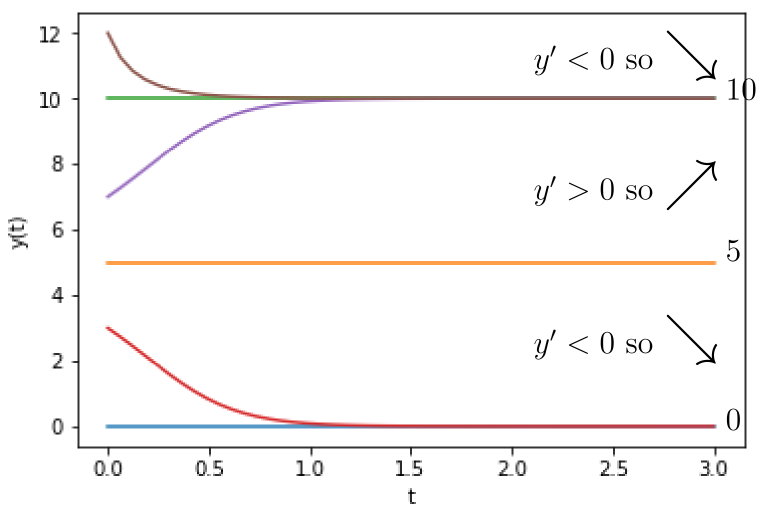
Solution: To find the equilibrium points we solve $.1y(y-5)(10-y) = 0$ which gives

The equilibrium points are $\underline{\ 0,\ 5,\ 10\ }$

To see which are stable and unstable plot $\dfrac{dy}{dt}$ as a function of y.



The graph is going downhill at 0 and 10 10 they are stable. At 5 the graph is going uphill and thus it is unstable. Alternatively we make the time series for y to see which points are stable:



Stable points are $\underline{\ 0,\ 10\ }$

Unstable points are $\underline{\ 5\ }$

(b) If $y(4) = 9$ what is $y'(4)$? *Solution:*

$$\begin{aligned} y'(4) &= .1y(4)(y(4) - 5)(10 - y(4)) \\ &= .1(9)(9 - 5)(10 - 1) \\ &= 3.6 \end{aligned}$$

□

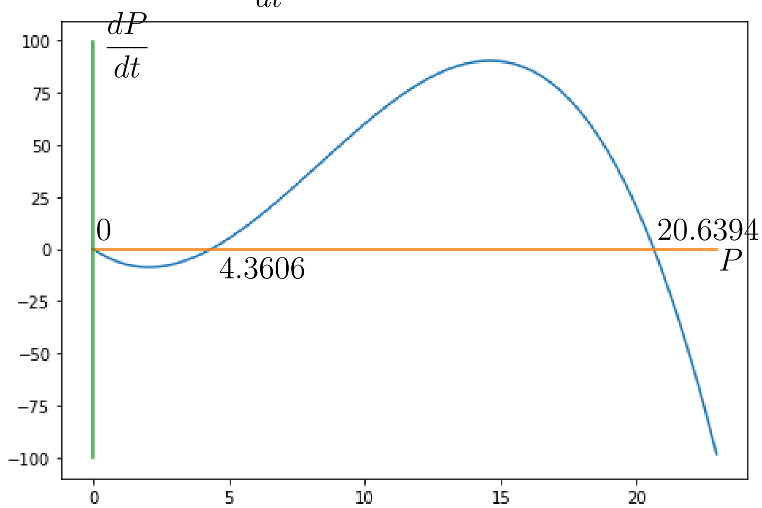
(c) If $y(0) = 7$ estimate $y(78)$. $\underline{y(78) \approx 10}$

This is because the solution will approach the stable equilibrium point $y = 10$. □

3. For the differential equation

$$\frac{dP}{dt} = -.1P^3 + 2.5P^2 - 9P$$

(a) Draw the graph of $\frac{dP}{dt}$ as a function of P with $0 \le P \le 23$.



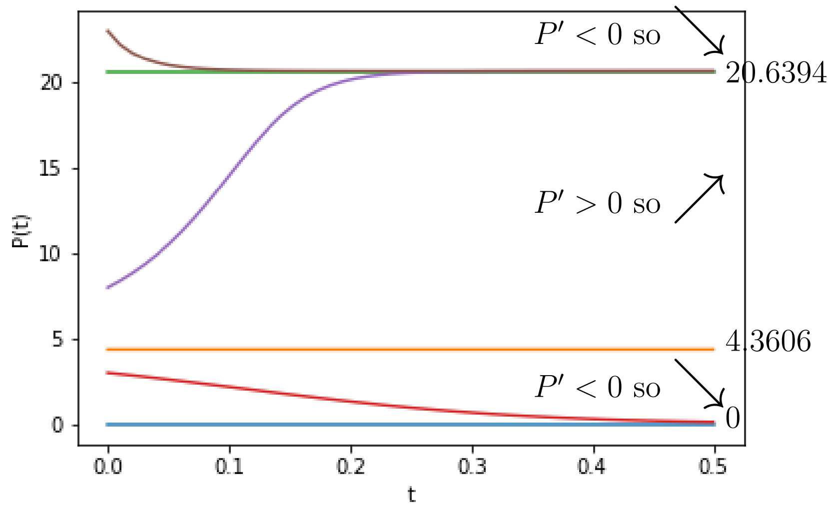
(b) Use your calculator to find the equilibrium points accurate to 3 decimal place and classify as to stable or unstable.

Equilibrium points are $\underline{0.000,\ 4.360,\ 20.639}$

Stable points are $\underline{0,\ 20.639}$

Unstable points are $\underline{4.360}$

As in the previous problem the stability can either be determined by looking to see if the graph is going up or down hill at the zeros, or by looking at the time series:



(c) If $P(0) = 10$ estimate $P(100)$. $P(100) \approx 20.639$

□

4. A pond has a population of duckweed growing in it. Due to goldfish in the pond feeding on it the intrinsic growth rate of the duckweed population is $-.2$ (lbs/week)/lb. The maintainer of the pond wish to have a stable duckweed population of 3 pounds of duck weed to provide shade for the fish. At what rate should she stock the pond?

Solution: Let $P(t)$ be the number of pounds of duckweed in the pond in week t and let S be the rate at which the pond will be stocked. Then the rate equation

$$\frac{dP}{dt} = -.2P + S$$

will hold. We wish $P = 3$ to be a stable equilibrium point. Putting this and $dP/dt = 0$ into the equation gives

$$0 = -.25(3) + S$$

which in turn gives that

$$S = .2(3) = .6 \text{ pounds/week}$$

as the stocking rate. □

5. A population of yeast is growing in a vat of water, malted barley, and hops. Let $P(t)$ be the number of grams of yeast in the vat after t days. Assume the yeast grows logistically with an inartistic growth rate of .15 (grams/day)gram and that the carrying capacity is 500 grams of yeast. Assume that the brewers start with 10 grams of yeast in the vat.

(a) What is the rate equation satisfied by P?

Solution: The logistic equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r is the intrinsic growth rate and K is the carrying capacity. So in our case the equation is

$$\frac{dP}{dt} = .15P\left(1 - \frac{P}{500}\right)$$

□

(b) Estimate the amount of yeast in the vat after 100 days?

Solution: By the 100th day the population will have leveled off at the carrying capacity and so $P(100) \approx 500$. □

(c) Estimate $P(2)$ by taking 20 Euler steps of size $h = .1$.

Solution: For this equation Euler's method will become

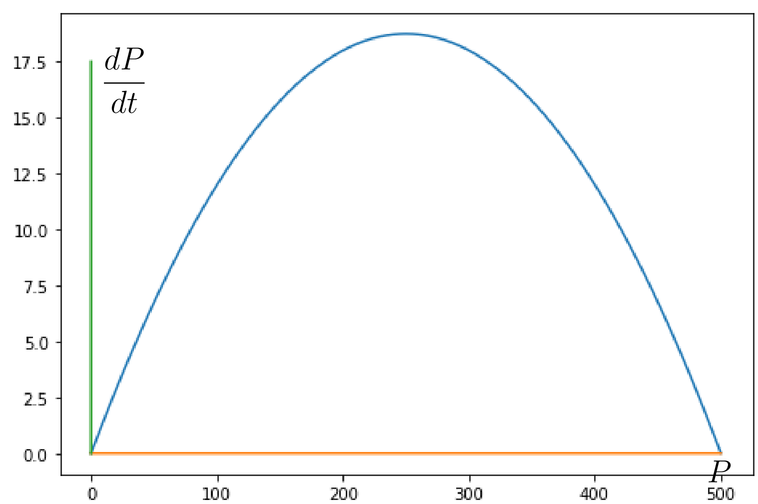
$$P_n = P_{n-1} + .15P_{n-1}\left(1 - \frac{P_{n-1}}{500}\right)(.1), \qquad P_0 = 10$$

Doing 20 steps of this gives $P_{20} = 13.377$. □

6. The brewers from the previous problem are raising the yeast to sell with beer making kits.

(a) What it the maximum rate they can harvest the yeast without kill off the population?

Solution: Plot $\frac{dP}{dt}$ as a function of P:



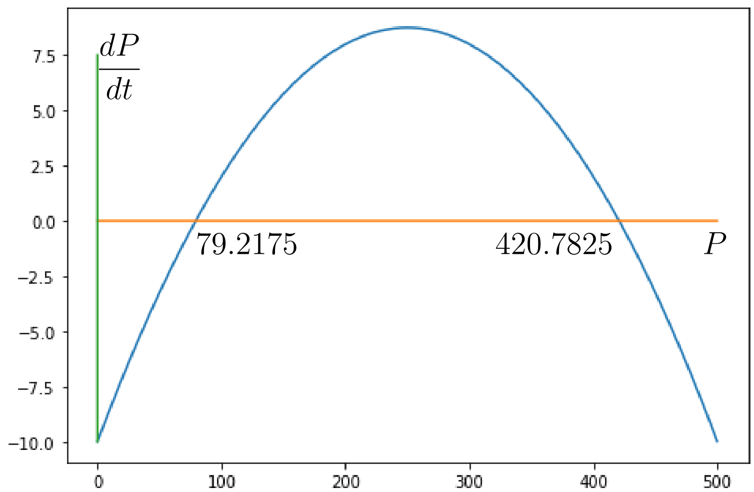
Now use the calculator to find that the maximum of $\frac{dP}{dt}$ is 18.75 (and this occurs when $P = 250$). So the maximum rate the yeast can be harvested is 18.75 grams/day. □

(b) Assume that they harvest the yeast at the rate of 10 grams/day. What happens to the yeast population?

Solution: The new rate equation for P is

$$\frac{dP}{dt} = .15P\left(1 - \frac{P}{500}\right) - 10.$$

This time the graph of $\frac{dP}{dt}$ as a function of P looks like



Form this we see that the stable equilibrium point is $P = 420.7825$ and so the yeast population will level off at 420.78.25 grams. □

7. Marigolds are an annual plant. Assume 25 marigolds are planted in a park and 5 years later there are 78 of them.

(a) What are is the growth ratio and per capita growth rate of the population?

Solution: Let N_t be the number of plants in year t and let λ be the growth ratio. Then, by the definition of the growth ratio

$$N_{t+1} = \lambda N_t.$$

We have seen that the solution to this is

$$N_t = N_0 \lambda^t.$$

We are given that $N_0 = 20$ and thus

$$N_t = 20\lambda^t.$$

As $N_5 = 78$ we get the equation

$$20\lambda^5 = 78$$

and therefore

$$\lambda = \left(\frac{78}{20}\right)^{\frac{1}{5}} = 1.31284.$$

Therefore the per capita growth rate is

$$r = \lambda - 1 = .31284$$

(b) Give a formula for the number of marigolds after t years.
Solution: This is

$$N_t = N_0 \lambda^t = 20(1.31284)^t.$$

(c) How long until there are 1,000 marigolds?
Solution: We wish to solve

$$N_t = 20(1.31284)^t = 1{,}000.$$

Dividing by 20 and taking the natural logarithm gives

$$t \ln(1.31284) = \ln(1{,}000/20)$$

and thus

$$t = \frac{\ln(1{,}000/20)}{\ln(1.31284)} = 14.37212$$

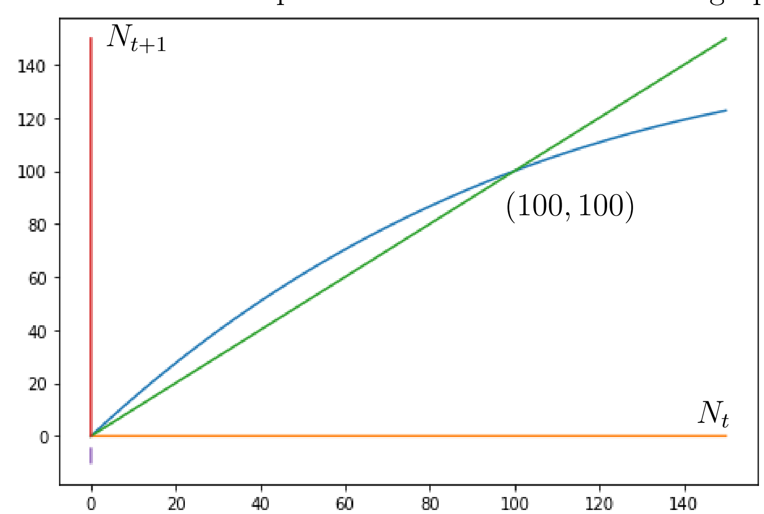
and so we need to round up to 15 years to get the 1,000 plants. □

8. For the discrete dynamical system

$$N_{t+1} = N_t e^{.4(1-N_t/100)}$$

(a) Graph N_{t+1} as a function of N_t with $0 \le N_t \le 150$ and draw the result here.

Solution: Here is the graph which also includes the graph of $N_{t+1} = N_t$ and shows the nonzero point in intersection of the two graphs.



(b) Find the equilibrium points of this system.

Solution: The equilibrium points are where the two graphs intersect. These are $N_* = 0$ and $N_* = 100$. $\square$

(c) If $N_0 = 50$ find N_1, N_2, and N_3.
Solution:

$$N_1 = 61.07013790800849$$
$$N_2 = 71.36035434069794$$
$$N_3 = 80.02195515288362$$