Math 172 Quiz 11. Name: Key

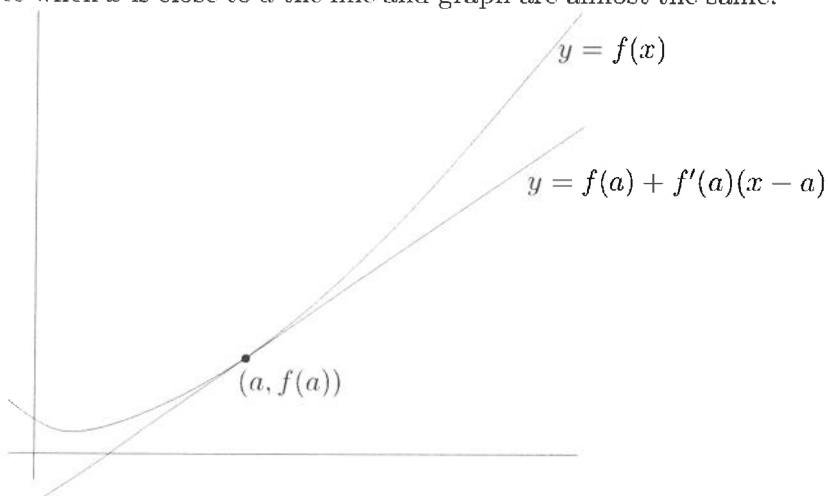
The tangent line to the function $y = f(x)$ at the point $(a, f(a))$ has equation

$$y - f(a) = f'(a)(x - a)$$

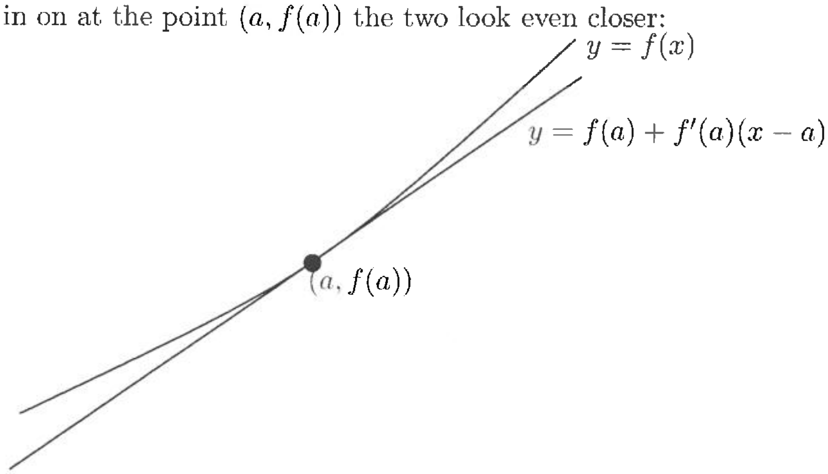
which can also be written as

$$y = f(a) + f'(a)(x - a).$$

I claim that when x is very close to a that $f(a) + f'(a)(x - a)$ is a very good approximation to $f(x)$. Here is the graph of a function and its tangent line at a point. Note when x is close to a the line and graph are almost the same.



If we zoom in on at the point $(a, f(a))$ the two look even closer:



Let us do a concrete example. The tangent line to

$$y = x^2$$

at the point $(2, 4)$ is

$$y = 4 + 4(x - 2)$$

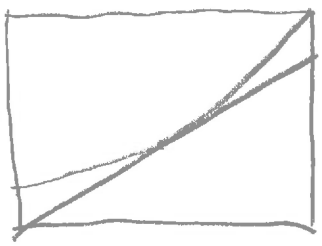
Problem 1. On your calculator let

\Y1=x^2

\Y2= 4 + 4(x-2)

Xmin=1 Xmax=3

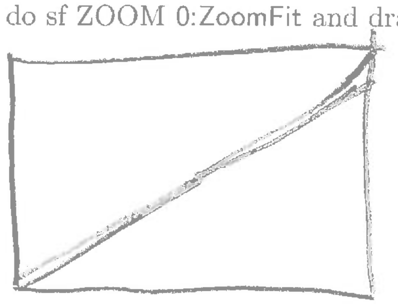
(a) Do ZOOM 0:ZoomFit and draw the result here.



(b) Part (a) we used a window of length $3 - 1 = 2$ center at 2. Let us zoom in a bit an use a window of length 1. So set

Xmin=1.5 Xmax=2.5

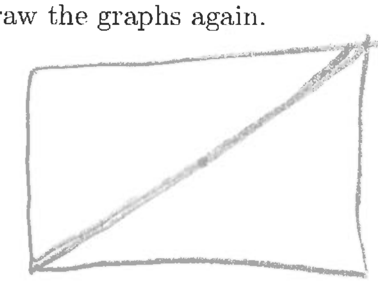
and again to a do sf ZOOM 0:ZoomFit and draw the result here.



(c) Now we use

Xmin=1.9 Xmax=2.1

and plot and draw the graphs again.

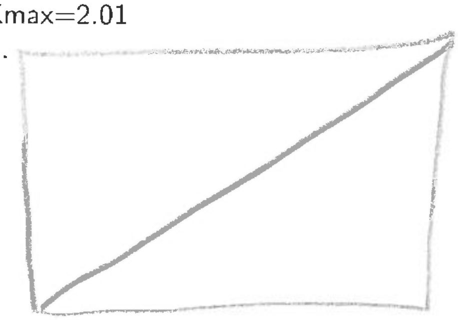


At this point the graphs should almost be on top of each other, but you should still be able to tell them apart.

(d) And one last time let

Xmin=1.99 Xmax=2.01

plot and draw again.



At this point, unless your calculator display is much better than mine, you should not be able to tell the difference between the two graphs. So working on the scale of x within a .01 of $a = 2$ for all practical purposes we can replace $y = f(x)$ by its tangent line approximation. $\square$

It is ofter useful to rewrite the basic approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

as follows. Let $x = a + h$ where we assume that h is close to 0. Then $x - a - = h$ and the tangent line approximation can be rewritten as

$$f(a+h) \approx f(a) + f'(a)h$$

Problem 2. Assume that we have a function $y = f(x)$ with $f(3) = 9$ and $f'(3) = 2$. Then approximate $f(3.1)$. To do this use the approximation $f(a+h) \approx f(a) + f'(a)h$ with $a = 3$ and $h = .1$ to get

$$f(3.1) = f(3 + .1) \approx f(3) + f'(3)(.1) = 9 + 2(.1) = 9.2$$

For this function estimate the following:

(a) $f(3.05)$ $\qquad f(3.05) \approx$ 9.1

$\approx f(3) + f'(3)(.05)$

$= 9 + 2(.05) = 9.1$

(b) $f(2.98)$ $\qquad f(2.98) \approx$ 8.96

$= f(3 + (-.02))$

$\approx f(3) + f'(3)(-.02) = 9 - 2(.02) = 8.96$

Just to change the notation yet again (I really don't do this just to confuse) let $y = y(t)$ be a function of t. Then for small h the tangent line approximation implies

$$y(t+h) \approx y(t) + y'(t)h.$$

We can not relate this to differential equations.

Problem 3. Consider the initial value problem

$$y' = 1 + y, \qquad y(1) = 2.$$

Let us use the tangent line approximation to estimate $y(1.1)$. From the tangent line approximation

$$y(1.1) \approx y(1) + y'(1)(.1)$$

Now use the equation to compute $y'(1)$.

$y'(1) =$ 3

$y'(1) = 1 + y(1) = 1 + 2 = 3$

Now use this to find the approximation to $y(1.1)$.

$y(1.1) \approx$ 2.3

$y(1.1) \approx y(1) + y'(1)(.1)$

$= 2 + 3(.1) = 2.3$

Problem 4. More generally assume that we have a differential equation

$$y' = f(y)$$

and assume we know the value of $y(t)$. Let h be a small number. Then the tangent line approximation to $y(t_h)$ is still

$$y(t + h) \approx y(t) + y'(t)h.$$

Use the equation $y = f(y)$ replace $y'(t)$ with $f(y(t))$.

$$y(t + h) \approx \underline{\;y(t) + y'(t)h\;}.$$

This is taking one step in Euler's method.