Mathematics 172

Quiz 18 **Name:**________________________________

You must show your work to get full credit.

In this quiz we see how to compute the stable age distribution exactly. Let us use the Leslie matrix from the last quiz, that is

$$L = \begin{bmatrix} 0.0 & 2.4 & 16.0 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \end{bmatrix}$$

Let

$$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be a vector that has the stable age distribution. Then, to use the terminology we have just introduced, we want to know when $\vec{N}$ is an ***eigenvector*** of L. That is when there is a number number λ (called the ***eigenvalue*** with $L\vec{N} = \lambda\vec{N}$.

1. Show that $L\vec{N} = \lambda\vec{N}$ is the same as This becomes

$$\begin{bmatrix} 2.4n_1 + 16n_2 \\ .1n_1 \\ .5n_2 \end{bmatrix} = \begin{bmatrix} \lambda n_1 \\ \lambda n_2 \\ \lambda n_3 \end{bmatrix}$$

which is equivalent to the three equations

$$2.4n_2 + 16n_3 = \lambda n_1 \tag{1}$$

$$.1n_1 = \lambda n_2 \tag{2}$$

$$.5n_2 = \lambda n_3. \tag{3}$$

2. Use equation (2) to show $n_2 = \dfrac{.1n_1}{\lambda}$. *Solution:* Just divide both sides of (2) by λ.

3. Use the last problem and equation (3) to show $n_3 = \dfrac{.5n_1}{\lambda} = \dfrac{(.5)(.1)n_1}{\lambda^2}$. *Solution:* First divide (3) by λ to get $n_3 = \dfrac{.5n_1}{\lambda}$. Now use the formula for n_2 from the last problem to get

$$n_3 = \frac{.5n_1}{\lambda} = \frac{.5\frac{.1n_1}{\lambda}}{\lambda} = \frac{(.5)(.1)n_1}{\lambda^2}.$$

4. Use the last two problems and equation (1) to show

$$\frac{2.4(.1)}{\lambda^2} + \frac{16(.5)(.1)}{\lambda^3} = 1.$$

Solution: First just plug in the formulas we have for n_2 and n_3 into (1) to get

$$2.4\frac{.1n_1}{\lambda} + 16\frac{(.5)(.1)n_1}{\lambda^2} = \lambda n_1.$$

Each term contains n_1 so we can divide this term out to get

$$2.4\frac{.1}{\lambda} + 16\frac{(.5)(.1)}{\lambda^2} = \lambda.$$

Now dividing by λ gives the desired equation. For use on the next problem note this can be simplified to

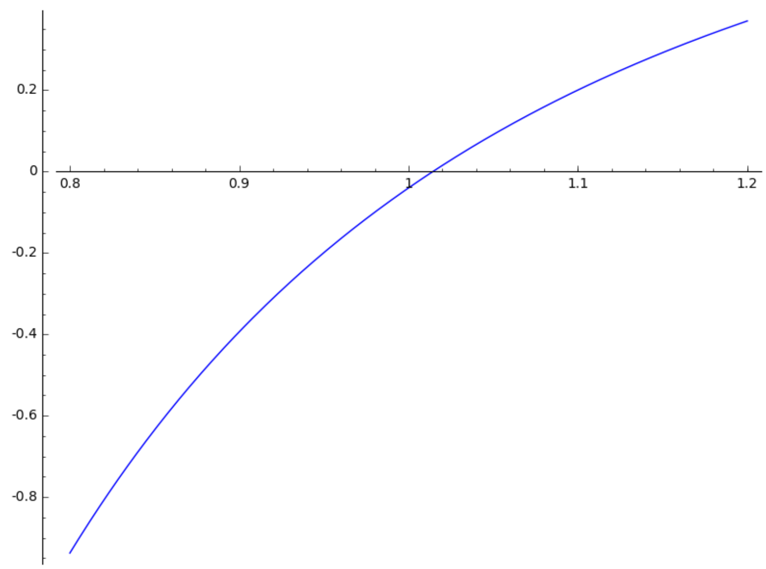
$$\frac{.24}{\lambda^2} + \frac{.8}{\lambda^3} = 1.$$

5. Use your calculator to solve this for λ.

Solution: In your calculator let

\Y$_1$ = 1 -.24/X^2 -.8/X^/3

Use Xmin = .8 and Xmax = 1.2 and then do a ZoomFit to get a graph that looks like



Then do 2nd CALC zero to find that

$$\lambda = 1.0142703$$

6. Now find n_2 and n_3 in terms of n_1. *Solution:* Using our formulas for n_2 and n_3 we find

$$n_2 = \frac{.1n_1}{\lambda} = .0986n_1$$

$$n_3 = \frac{(.1)(.5)n_1}{\lambda^2} = .0486n_1$$

7. Finally give the stable age distribution.

Solution: The total in all the classes is

$$n = n_1 + n_2 + n_3 = n_1 + .0986n_1 + .0486n_1 = 1.1472n_1$$

Thus we have

$$\text{Proportion in Stage 1} = \frac{n_1}{n} = \frac{n_1}{1.1472n_1} = .872 = 87.2\%$$
$$\text{Proportion in Stage 2} = \frac{n_1}{n} = \frac{.0986n_1}{1.1472n_1} = .086 = 8.6\%$$
$$\text{Proportion in Stage 3} = \frac{n_1}{n} = \frac{.0486n_1}{1.1472n_1} = .042 = 4.2\%$$

8. Finally, what are the growth ratio and the per capita growth rate?.

Solution: The growth ratio is the number λ we found above: $\lambda = 1.0142703$. The per capita is $r = \lambda - 1 = .0142703$.

Now let us do this all over again for a general Leslie matrix,

$$L = \begin{bmatrix} f_1 & f_2 & f_3 \\ p_1 & 0 & 0 \\ 0 & p_2 0 & \end{bmatrix}$$

Let

$$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be an eigenvector with eigenvalue λ. Then

$$L\vec{N} = \lambda\vec{N}.$$

9. Show this leads to the equations

$$f_1 n_1 + n_2 f_2 + n_3 f_3 = \lambda n_1 \tag{4}$$
$$p_1 n_1 = \lambda n_2 \tag{5}$$
$$p_2 n_2 = \lambda n_3 \tag{6}$$

Solution: This is either done by matrix multiplication or by using a loop diagram as we have done in class many times.

10. Do calculations as above to show

$$n_2 = \frac{p_1 n_1}{\lambda} \quad \text{and} \quad n_3 = \frac{p_1 p_2 f_3}{\lambda^2}.$$

Solution: Dividing equation (5) by λ gives

$$n_2 = \frac{p_1 n_1}{\lambda}$$

which is one of the two equation we want. To get the section divide equation (6) by λ to get

$$n_3 = \frac{p_2}{\lambda} n_2.$$

Now use the formula for n_2 we have just found to get

$$n_3 = \frac{p_2}{\lambda}\left(\frac{p_1 n_1}{\lambda}\right) = \frac{p_1 p_2 n_1}{\lambda^2}$$

as required.

11. Use these formulas for n_1 and n_2 in equation (4) to get

$$f_1 + \frac{p_1 f_2}{\lambda^2} + \frac{p_1 p_2 f_3}{\lambda^3} = \lambda.$$

Solution: Do the substations for n_2 and n_3 in (4) gives

$$f_1 n_1 + \left(\frac{p_1 n_1}{\lambda}\right) f_2 + \left(\frac{p_1 p_2 n_1}{\lambda^2}\right) f_3 = \lambda n_1$$

Dividing by the common factor gets rid of the n_1 factors to give

$$f_1 + \left(\frac{p_1}{\lambda}\right) f_2 + \left(\frac{p_1 p_2}{\lambda^2}\right) f_3 = \lambda$$

which simplifies down to the equation we want.

12. Divide the last equation by λ to get

$$\frac{f_1}{\lambda} + \frac{p_1 f_2}{\lambda^2} + \frac{p_1 p_2 f_3}{\lambda^3} = 1.$$

This is the ***Euler-Lotka equation***. It has a unique positive solution, which is the growth of the stable age distribution of the population.

Solution: Just divide by λ.

13. Now that we have λ and also the formulas for n_2 and n_3, use this to find the stable age distribution.

Solution: We have that the total (the sum of the entries in $\vec{N}$) is

$$\begin{aligned} n &= n_1 + n_2 + n_3 \\ &= n_1 + \frac{p_1 n_1}{\lambda} + \frac{p_1 p_2 n_1}{\lambda^2} \\ &= n_1 \left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right) \end{aligned}$$

Therefore we have

$$\text{Proportion in Stage 1} = \frac{n_1}{n} = \frac{n_1}{n_1\left(1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}\right)} = \frac{1}{1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}}$$

$$\text{Proportion in Stage 2} = \frac{n_2}{n} = \frac{\frac{p_1n_1}{\lambda}}{n_1\left(1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}\right)} = \frac{\frac{p_1}{\lambda}}{1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}}$$

$$\text{Proportion in Stage 3} = \frac{n_3}{n} = \frac{\frac{p_1n_1}{\lambda}}{n_1\left(1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}\right)} = \frac{\frac{p_1p_2}{\lambda^2}}{1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}}$$

This may be easier to take in when written in vector notation:

$$\text{Vector of the stable age distribution} = \frac{1}{1+\frac{p_1}{\lambda}+\frac{p_1p_2}{\lambda^2}} \begin{bmatrix} 1 \\ \\ \frac{p_1}{\lambda} \\ \\ \frac{p_1p_2}{\lambda^2} \end{bmatrix}$$