Mathematics 172

Quiz 2 Name: Key

You must show your work to get full credit.

Something that is well modeled by our basic growth equation

$$\frac{dA}{dt} = rA$$

is radioactive decay. In this case the relative growth rate

$$r = \frac{1}{A}\frac{dA}{dt}$$

is roughly the proportion of the substance that decays in a unit period of time.

1. The isotope 238 of uranium (denoted ^{238}U) has a half life of 4.468 billion years.

(a) Use the half life to find the relative growth r and include units in your answer.

$A(t) = A(0)e^{rt}$

we wish to solve

$A(4.468) = A(0)e^{4.468r} = .5A(0)$

$e^{4.468r} = .5$

$4.486r = \ln(.5)$

$r = \frac{\ln(.5)}{4.486} = -0.15451$

$r =$ -0.1541 / billion years

(b) Starting with a sample of pure ^{238}U it will decay in to lead[1] Thus it is possible to tell how old a sample of ^{238}U is by measuring the percent of the ^{238}U that is left in the same, which in turn can be measured by looking at the ratio of ^{238}U and lead. Assume that a rock has 45% of its original ^{238}U. How old is it?

$A(t) = A(0)e^{-.15451t}$

This time we solve

$A(0)e^{-.15451t} = .45A(0)$

$e^{-.15451t} = .45$

$-.15451t = \ln(.45)$

$t = \frac{\ln(.45)}{-.15451} = 5.1680$

The age is 5.1680 billion years

[1]It is somewhat more complicated than this as there are several intermediate elements involved in the ^{238}U becoming lead.

2. This problem is partly to show just how fast exponential function grow. Under ideal conditions the bacterium E. coli will double every 1/3 hours. A single E. coli weighs 10^{-15} kg.

(a) We start with a colony of a single E. coli. Let $W(t)$ be the wight in kilograms of the colony after t hours. Assume unconstrained population growth this will satisfy

$$\frac{dW}{dt} = rW$$

for some constant r. Find r and give its units. $r =$ 2.07944(kg/h)/kg

$W(t) = 10^{-15} e^{rt}$ so solve

$W(\frac{1}{3}) = 10^{-15} e^{\frac{r}{3}} = 2(10^{-15})$

$e^{\frac{r}{3}} = 2$

$\frac{r}{3} = \ln(2)$

$r = 3\ln(2) = 2.07944$

(b) Give a formula for $W(t)$. $W(t) = 10^{-15} e^{2.07944t}$ kg

(c) The weight of the Earth is 6.0×10^{24}kg. How long before the colony of E. coli has it weight equal to that of the Earth?

Solve

$10^{-15} e^{2.07944t} = 6.0 \times 10^{24}$ $t =$ 44.047 hours (less than 2 days)

$e^{2.07949t} = 6.0 \times 10^{24} \cdot 10^{15} = 6.0 \times 10^{39}$

$2.07944t = \ln(6.0 \times 10^{39})$

$t = \ln(6.0 \times 10^{39}) / 2.07944$

$= 44.047$