Mathematics 172

Quiz 24 Name: Key

You must show your work to get full credit.

We are look at Euler's method for a system of rate equations with initial conditions:

$$\frac{dx}{dt} = f(x,y), \qquad x(t_0) = x_0$$
$$\frac{dy}{dt} = g(x,y), \qquad y(t_0) = y_0.$$

It will be based on the basic approximations that we already know and love:

$$x(t+h) \approx x(t) + x'(t)h$$
$$y(t+h) \approx y(t) + y'(t)h$$

which holds when h is small.

To start we choose a small number h, the ***step size***. Let $k \geq 0$ be an integer and assume that we have computed t_k, x_k, and y_k in such a way that

$$x_k \approx x(t_k)$$
$$y_k \approx y(t_k).$$

To be explicit about what this notation means here x_k is our approximation to the value of the true solution $x(t)$ at the point where $t = t_k$. Then by our basic approximations we have

$$x(t_{k+1}) = x(t_k + h) \approx x(t_k) + x'(t_k)h \approx x_k + x'(t_k)h$$
$$y(t_{k+1}) = y(t_k + h) \approx y(t_k) + y'(t_k)h \approx y_k + y'(t_k)h$$

But from the differential equations for x and y we have

$$x'(t_k) = f(x(t_k), y(t_k)) \approx f(x_k, y_k),$$
$$y'(t_k) = g(y(t_k), y(t_k)) \approx g(x_k, y_k).$$

Putting these approximation together gives

$$x(t_{k+1}) \approx x_k + f(x_k, y_k)h,$$
$$y(t_{k+1}) \approx y_k + g(x_k, y_k)h.$$

So to summarize here is Euler's method for the system:

Initial Step: Set

$$t_0 = t_0$$
$$x_0 = x_0$$
$$y_0 = y_0.$$

Euler Step from k to $k+1$:

$$t_{k+1} = t_k + h$$
$$x_{k+1} = x_k + f(x_k, y_k)h$$
$$y_{k+1} = y_k + g(x_k, y_k)h$$

It is not hard to see that after n steps we have that

$$t_n = t_0 + nh.$$

Then x_n and y_n will be good approximations to the true values $x(t_n)$ and $y(t_n)$.

Let us now do an example on the calculator. As a sample system we use

$$\frac{dx}{dt} = 2x - 3y \qquad x(0) = 4$$
$$\frac{dy}{dt} = -x + 2y \qquad y(0) = 1$$

and we will approximate $x(2)$ and $y(2)$ by taking 20 steps of size $h = .1$. The scheme for this is

$$t_0 = 0$$
$$x_0 = 4$$
$$y_0 = 1$$

and taking an Euler step looks like

$$t_{k+1} = t_k + .1$$
$$x_{k+1} = x_k + (2x_k - 3y_k)(.1)$$
$$y_{k+1} = y_k + (-x_k + 2y_h)(.1)$$

To set the calculator up to deal with this go to the [MODE] screen and edit to look like

[NORMAL] SCI ENG
[FLOAT] 0 1 2 3 4 5 6 7 8 9
[RADIAN] DEGREE
FUNC PAR POL [SEQ]
[CONNECTED] DOT
[SEQUENTIAL] SIMUL
[REAL] a+bi re^θi
[FULL] HORIZ G-T

Store the value of the step size in the H register: at the main screen and type .1 STO ALPHA H
Press 2ND TABLESET and edit until it looks like

TABLE SETUP
TblStart=0
ΔTbl=1
Indpnt : [Auto] Ask
Depend: [Auto] Ask

To enter the equations go to the Y= window and (where we will use u for x and v for y) and enter:

Plot1 Plot2 Plot2
nMin=0
\u(n)=u(n-1)+(u(n-1)-3v(n-1))H
u(nMin)=4
\v(n)=v(n-1)+(-u(n-1)+2v(n-1))H
v(nMin)=10
\w(n)=
w(nMin)=

Here n is entered with the [X,T,θ, n] bottom, u is entered with 2ND u (which is the 7 key), v is entered with 2ND v (which is above the 8 key), and H is entered with ALPHA H.

Now 2ND TABLE will give you the first first several values for x_k and y_k. To easily get access to more values go back to 2ND TABLESET and edit until it looks like:

TABLE SETUP

TblStart=0

ΔTbl=1

Indpnt : Auto [Ask]

Depend: [Auto] Ask

and you can now get that

$$x(2) \approx x_{20} = 649.49$$
$$y(2) \approx y_{20} = -369.4$$

1. (a) With the same system use 40 steps of size $h = .05$ to approximate $x(2)$ and $y(2)$.

$x(2) \approx x_{40} =$ 1068.6 $\qquad y(2) \approx y_{40} =$ -611.3

(b) Get a still better approximation of $x(2)$ and $y(2)$ by taking 200 steps of size $h = .01$.

$x(2) \approx x_{200} =$ 1731.5 $\qquad y(2) \approx y_{200} =$ -994

2. For the initial value problem

$$\frac{dy}{dt} = .05x\left(\frac{10 - x - .2y}{10}\right) \qquad x(0) = 3$$
$$\frac{dy}{dt} = .03y\left(\frac{20 - .5x - y}{20}\right) \qquad y(0) = 2$$

(a) Use 20 steps of size $h = .1$ to estimate $x(2)$ and $y(2)$.

$x(2) \approx x_{20} =$ 3.201 $\qquad y(2) \approx y_{20} =$ 2.1008

(b) Get a better approximation of $x(2)$ and $y(2)$ by taking 40 steps of size $h = .05$.

$x(2) \approx x_{40} =$ 3.2011 $\qquad y(2) \approx y_{40} =$ 2.1008

(c) Get a still better approximation of $x(2)$ and $y(2)$ by taking 200 steps of size $h = .01$.

$x(2) \approx x_{200} =$ 3.2011 $\qquad y(2) \approx y_{200} =$ 2.009

So we have computed x(2) and y(2) to four decimal places.