Mathematics 172

Quiz 30 Name: Key

You must show your work to get full credit.

Consider a population of size N which at any time as two classes. Those that are infected with some parasite (such as lice) and those that are not infected but are susceptible to being infected. Let $S(t)$ be the number of susceptibles at time t and $I(t)$ the number of infected at time t. Then

$$S(t) + I(t) = N.$$

We assume that infected individuals can give the infection to susceptible individuals and that the likely hood of a susceptible individuals being infected is proportional to its number of contacts with a infected individuals. We also assume that a unit time period a proportion, p, of the infecteds get rid of the parasite and move into the population of susceptibles.

A model for this system is

$$\begin{aligned} \frac{dS}{dt} &= -bSI + pI \\ \frac{dI}{dt} &= bSI - pI \end{aligned} \tag{1}$$

1. Use these equation to show that $S+I$ is constant by showing $\frac{d(S+I)}{dt} = 0$. (This can be viewed as redundant as we are assuming that $S + I = N$ and N is constant. But consider this as a check that our model is consistent with the facts.)

$$\frac{d(S+I)}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = -bSI + pI + bSI - pI = 0$$

2. In the equation $S + I = N$ solve for I and use the result in (1) to get the equation

$$\frac{dS}{dt} = (N - S)(-bS + p).$$

$I = (N-S)$, so

$$\begin{aligned} \frac{dS}{dt} &= -bSI + pI \\ &= I(-bS + p) \\ &= (N-S)(-bS+p) \end{aligned}$$

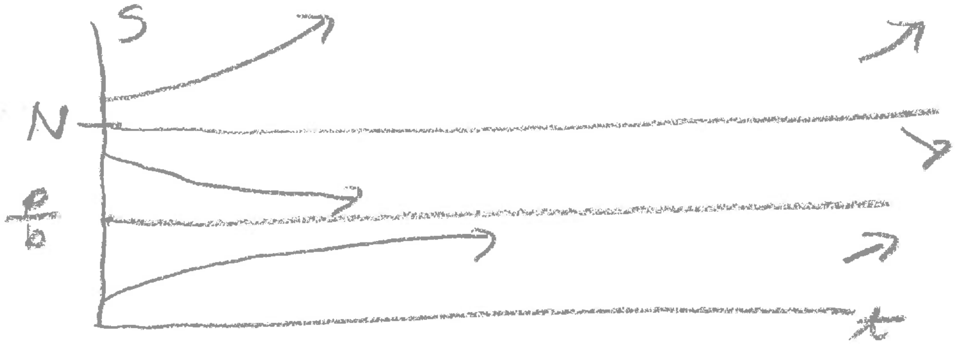
3. Show the equilibrium points of this rate equation for S are $S = N$ and $S = p/b$.

Solve $\frac{dS}{dt} = (N-S)(-bS+p) = 0$ for S.

$N - S = 0 \Rightarrow S = N$

$-bS + p = 0 \Rightarrow S = \frac{p}{b}$

4. (a) Show that if $N > p/b$, then N is unstable and p/b is stable. Do this by drawing pictures of what the time series look like.



$\frac{dS}{dt} = (N-S)(-bS+p)$

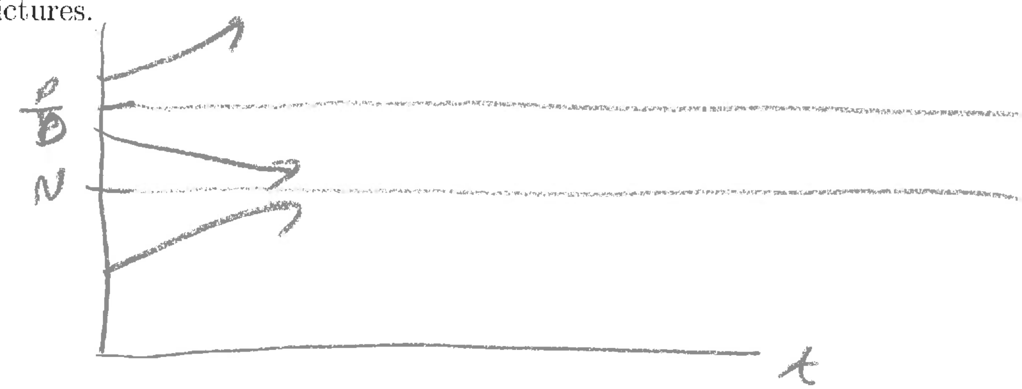
Positive for lower S

Picture shows $\frac{p}{b}$ stable, N unstable

(b) What does this mean in terms of the long term behavior of the infection?

If $0 < S(0) < N$, then $S(\text{big}) \approx \frac{p}{b}$. That is the system stabilizes with $S = \frac{p}{b}$ and $I = N - \frac{p}{b}$. So the infection keeps moving through the population.

5. (a) Show that if $N < p/b$, then N is stable and p/b is unstable. Again do this by drawing pictures.



$\frac{dS}{dt} = (N-S)(-bS+p)$

positive for large S

Picture shows N stable and $\frac{p}{b}$ unstable.

(b) What does this mean in terms of the long term behavior of the infection?

If $0 < S(0) < N$, then $S(\text{big}) \approx N$. That is in this case the infection dies off.