Mathematics 172 Test 3 Name: Key

You are to use your own calculator, no sharing.
Show your work to get credit.

1. (15 points) Let $x(t)$ and $y(t)$ satisfy the differential equations

$$\frac{dx}{dt} = .5x - .3y$$
$$\frac{dy}{dt} = .1x + .2y$$

(a) If $x(5) = 3.2$ and $y(5) = 2.1$ compute $x'(t)$ and $y'(t)$.

$x'(5) =$ $.97$

$x'(5) = .5\,x(5) - .3\,y(5)$
$= .5\,(3.2) - .3\,(2.1)$
$= .97$

$y'(t) =$ $.74$

$y'(5) = .1\,x(5) + .2\,y(5)$
$= .1(3.2) + .2(2.1)$
$= .75$

(b) Estimate $x(5.2)$ and $y(5.2)$.

$x(5.2) \approx$ 3.394

$x(5.2) \approx x(5) + x'(5)(.2)$
$= 3.2 + (.97)(.2)$
$= 3.394$

$y(5.2) \approx$ 2.248

$y(5.2) \approx y(5) + y'(5)(.2)$
$= 2.1 + (.74)(.2)$
$= 2.248$

2. (10 points) For the predator-victim system

$$\frac{dV}{dt} = .05V\left(1 - \frac{V}{400}\right) - .02VP$$
$$\frac{dP}{dt} = -.5P + .001VP = P(-.5 + .001V)$$

explain why the predators will die off.

From the $\frac{dP}{dt}$ equation we see if $V < \frac{.5}{.001} = 500$ Then P is decreasing. (i.e. $V = 500$ is the "famine line"). The carry capacity of the victums is $K = 400$ which is less than the famine line. So there are not enough victums to feed the predators and so the predators die off.

3. (25 points) For the predator-victim system

$$\frac{dV}{dt} = .04V\left(1 - \frac{V}{800}\right) - .01VP = V\left(.04\left(1-\frac{V}{800}\right) - .01P\right)$$

$$\frac{dP}{dt} = -.4P + .02VP = P(-.4 + .02V)$$

(a) If there are no predators what is the carrying capacity of the victims?

$K =$ 800

(b) What is the equation of the "famine line" for the predators? That is the line $V =$ constant that makes $dP/dt = 0$.

Set $\frac{dP}{dt} = 0$ to set $P = 0$ and $-.4 + .02V = 0$

i.e. $V = \frac{.4}{.02} = 20$

The equation is $V = 20$

(c) Find the equilibrium points. The equilibrium points are: (0,0), (800,0), (20,3.9)

(0,0), (800,0)

$V = 20$ makes $\frac{dP}{dt} = 0$. use this in $\frac{dV}{dt} = 0$ to set $.04\left(1 - \frac{20}{800}\right) - .01P = 0$

$P = \frac{.04}{.01}\left(1 - \frac{20}{800}\right) = 3.9$

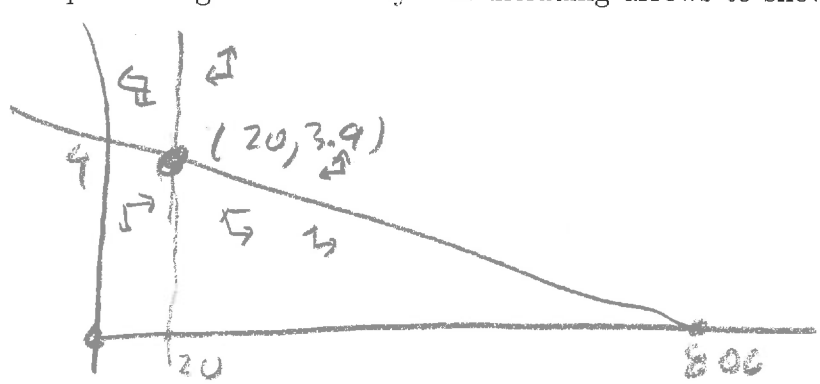
(d) For each of the equilibrium points explain what they mean biologically.

(0,0) No predators or victims and nothing happens

(800,0). No predators and the victim population stabilizes at its carrying capacity

(20,3.9). There are enough victims to feed the predators, and the populations stabilize at $V = 2$, $P = 3.9$

(e) Draw the phase diagram for the system including arrows to show the direction that points are moving.



4. (20 points) For a SIR model of the spread of an infection, let S_t be the number of susceptibles on day t, I_t the number of infecteds on day t, and R_t the number of recovered (or removed) on day t. Assume

$$\Delta S = -.002SI$$
$$\Delta I = .002SI - .1I$$
$$\Delta R = .1I$$

(a) What is the average length of an infection?

The average length is $\frac{1}{.1} = 10$ days

(b) Assume that in a population of 60 hogs in on a farm is modeled by this system and that on day 0 of an infection $S_0 = 30$ and $I_0 = 10$. Find the following:

$S_1 =$ 29.4 $\quad I_1 =$ 9.6 $\quad R_1 =$ 21

$\Delta S = -.002(30)(10) = -.6$

$\Delta I = .002(30)(10) - .1(10) = -.4$

$\Delta R = .1(10) = 1$

$R_0 = 60 - S_0 - I_0 = 60 - 30 - 10 = 20$

$S_1 = S_0 + \Delta S = 30 - .6 = 29.4$

$I_1 = I_0 + \Delta I = 10 - .4 = 9.6$

$R_1 = R_0 + \Delta R = 20 + 1 = 21$

(c) With these values of S_0, I_0, R_0 is the infection increasing or decreasing? Explain how you arrived at your answer.

$\Delta I = -.6$ so the infection is decreasing.
And since $\Delta I = I(.002S - .1)$ and $S < \frac{.1}{.002} = 50$
it will continue to decrease.

5. (10 points) A 12 inch largemouth bass weights 14 oz. Let L be the length of a bass and W its weight. Assume that bass stay the same shape as they grow.

(a) Give a formula for W in terms of L. $\quad W =$ $.0081 L^3$

Weight is proportional to volume
which is proportional to $(\text{length})^3$.
so $W = k L^3$ for some k
when $L = 12$, $W = 14$ so $14 = k(12)^3$ $\Rightarrow k = \frac{14}{(12)^3} = .0081$

(b) How much does a 18 in ball weigh?

Let $L = 18$ in the
formula above
$W = (.0081)(18)^3$
$= 47.2$

The weight is 47.2 oz

6. (15 points) Assume we have a population of birds such that is infected with a parasite that only stays on the birds for a short period of time. In a population of 200 birds let $S(t)$ be the number of susceptible birds on day t, and $I(t)$ the number of infected birds on day t. Assume that these satisfy the system of rate equations

$$\frac{dS}{dt} = -.002SI + .2I$$
$$\frac{dI}{dt} = .002SI - .2I$$

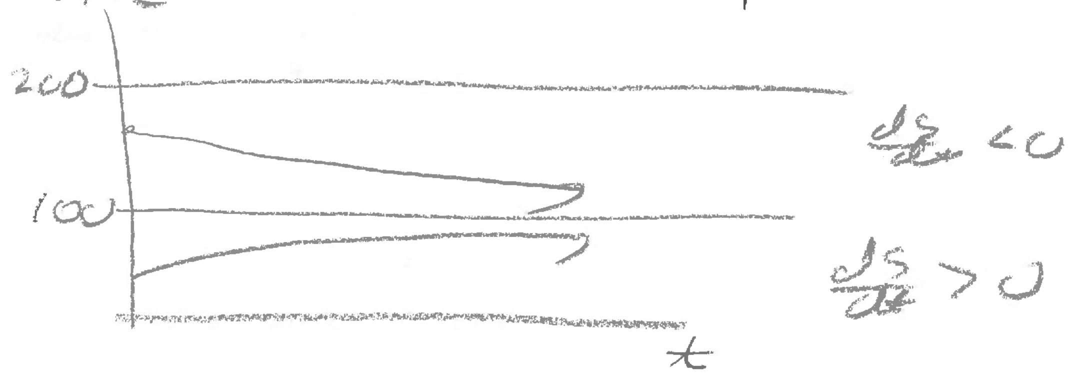
What is the long term behavior of this system. *Hint:* Solve for I in the equation $S + I = 200$ and use this in the first rate equation to get an equation that only involves S.

From $S+I=200$ we have $I=200-S$

Then

$$\frac{dS}{dt} = I(-.002S+.2)$$
$$= (200-S)(-.002S+.2)$$

Setting $\frac{dS}{dt}=0$ gives $S=200$ and $S=\frac{.2}{.002}=100$ as equilibrium points. The time series looks like



So things stabilize with $S=100$, $I=200-S=200-100=100$.

7. 5 free points.

Have a good thanksgiving.