Mathematics 242 Homework.

The main topic we covered in class today was homogeneous second order with constant coefficients. That is equations of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$. The ***characteristic equation*** of this differential equation is the algebraic equation

$$ar^2 + br + c = 0.$$

This is quadratic equation in r and can be solved by factoring, completing the square, or the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What we saw today was

Theorem 1. *If the roots r_1 and r_2 of the characteristic equation are real and distinct (that is $r_1 \neq r_2$) then the general solution to*

$$ay'' + by' + cy = 0$$

is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$ □

Example 2. Find the general solution to the equation

$$y'' + 3y' - 10y.$$

In this case the characteristic equation is

$$r^2 + 3r - 10 = (r + 5)(r - 2)$$

so that the characteristic roots are $r_1, r_2 = -5, 2$ and thus the general solution is

$$y = c_1 e^{-5x} + c_2 e^{2x}.$$ □

Example 3. Find the general solution to

$$y'' + 2y' - 2y = 0.$$

In principle this is not any harder than the previous example, other than the characteristic equation

$$r^2 + 2r - 2 = 0$$

does not factor nicely. So we use the quadratic formula to get

$$r_1, r_2 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}.$$

Therefore the general solution is

$$y = c_1 e^{(-1-\sqrt{3})x} + c_2 e^{(-1+\sqrt{3})x}.$$ □

Problem 1. Find the general solutions to the following:
(a) $y'' - 9y' + 20y = 0$
(b) $3y'' - 8y' + 4y = 0$
(c) $y'' - ky = 0$ where $k > 0$ is a constant.
(d) $y'' + 2ky' - y = 0$ where k is a constant. □

We also saw that if the values of y and y' are given at some point, then we can solve for c_1 and c_2 in the general solution.

Example 4. For the equation

$$r^2 + 3r - 10 = (r+5)(r-2)$$

find the solution with $y(0) = 5$ and $y'(0) = 4$. We have already seen that the general solution to this equation is

$$y = c_1 e^{-5x} + c_2 e^{2x}.$$

Then the derivative is

$$y' - 5c_1 e^{-5x} + c_2 e^{2x}.$$

Then we want to choose c_1 and c_2 so that

$$y(0) = c_1 + c_2 = 5$$
$$y'(0) = -5c_1 + 2c_2 = 4$$

(where we have used $e^0 = 1$). Solving (I leave the algebra to you) we get

$$c_1 = \frac{6}{7}, \qquad c_2 = \frac{29}{7}$$

and therefore the solution we are after is

$$y = \frac{6}{7}e^{-5x} + \frac{29}{7}e^{2x}.$$ □

Example 5. Find the solution to

$$y'' + 3y' + 2 = 0$$

with $y(1) = 2$ and $y'(1) = -7$. The characteristic equation is

$$r^2 + 3r + 2 = (r+2)(r+1) = 0$$

so the characteristic roots are -1 and -2, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

and its derivative is

$$y = -c_1 e^{-x} - 2c_2 e^{-2x}.$$

We need to solve for c_1 and c_2 in

$$y(1) = c_1 e^{-1} + c_2 e^{-2} = 2$$
$$y'(1) = -c_1 e^{-1} - 2e^{-2} = -7.$$

I again leave the algebra to you in showing

$$c_1 = -3e, \qquad c_2 = 5e^2$$

and therefore the solution we are after is

$$y = -3ee^{-x} + 5e^2e^{-2} = -3e^{-(x-1)} + 5e^{-2(x-1)} \qquad \square$$

Problem 2. Solve the following initial value problems:
(a) $y'' + 7y' + 12y = 0$, with $y(0) = 4$, and $y'(0) = -3$.
(b) $y'' + 7y' + 12y = 0$, with $y(3) = 4$, and $y'(3) = -3$.
(c) $y'' - 4y' - 2y = 0$, $y(0) = 1$, and $y'(0) = -3$.
(d) $y'' - ky = 0$ with $y(0) = y_0$, and $y'(0) = y_1$ and $k > 0$ is a constant. $\square$

The main general result for linear homogeneous second order differential equations is

Theorem 6. *Let $A(x)$, $B(x)$, and $C(x)$ be defined on a interval (a, b) with $A(x) \neq 0$ on this interval. Let y_1 and y_2 be linearly independent solutions to*

$$L(y) = A(x)y'' + B(x)y' + C(x)y = 0.$$

Then the general solution to the equation is

$$y = c_1y_1 + c_2y_2$$

where c_1 and c_2 are constants. $\square$

It is always the case that linearly independent solutions y_1 and y_2 exist, which is also an important part of the theory.

If u and v are differentiable functions then the ***Wronskian*** of u and v is

$$W = W[u, v] = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v.$$

The reason we care about this is that it gives an easy check to see if two functions are linearly independent.

Theorem 7. *Let u and v be two differentiable on an interval and assume that the Wronskian $W = W[u, v] \neq 0$ for at least one point. Then u and v are linearly independent.*

Proof. We did this in class. $\square$

Example 8. Show the two functions e^{2x} and e^{-x} are linearly independent.

In is enough to show that the Wronskian is nonzero.

$$\begin{aligned} W[e^{2x}, e^{-x}] &= \begin{vmatrix} e^{2x} & e^{-x} \\ (e^{2x})' & (e^{-x})' \end{vmatrix} \\ &= \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} \\ &= -3e^{x} \\ &\neq 0 \end{aligned}$$

and therefore the functions are linearly independent. $\square$

Problem 3. Let r_1 and r_2 be real numbers with $r_1 \neq r_2$. Compute the Wronskian $W[e^{r_1}, e^{r_2}]$ and use it to show that $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent. □

Problem 4. Compute the Wronskian of $\cos(x)$ and $\sin(x)$ and use it to show $\cos(x)$ and $\sin(x)$ are linearly independent. □

Problem 5. Generalizing the previous problem, let α and β be real numbers with $\beta \neq 0$. Compute the Wronskian of $y_1 = e^{\alpha x}\cos(\beta)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. Use this to conclude these two function are linearly independent. □

Problem 6. Let r be a real number and $y_1 = e^{rx}$ and $y_2 = xe^{rx}$. Compute the Wronskian of y_1 and y_2 and use it to show y_1 and y_2 are linearly independent.

We now summarize what happens for the constant coefficient second order linear homogeneous equation. Let a, b, and c be constants with $a \neq 0$. We wish to solve

$$ay'' + by' + cy = 0.$$

The ***characteristic equation*** for this is

$$ar^2 + br + c = 0$$

and its roots are

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider.

(i) $r_1 \neq r_2$ are real (i.e. $b^2 - 4ac > 0$). Then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 i x} + c_2 e^{r_2 i x}.$$

(ii) $r_1 = r_2$ are real (i.e. $b^2 - 4ac = 0$). Then let $r = r_1 = r_2$. he general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

(iii) r_1 and r_2 are complex (i.e. $b^2 - 4ac < 0$). Then let $r = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ he general solution to $ay'' + by' + cy = 0$ is

$$\begin{aligned} y &= c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\cos(\beta x) \\ &= e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right). \end{aligned}$$

Problem 7. (a) Find the general solution to $y'' + 4y' + 4y = 0$ and then find the solution to the initial value problem of this equation with $y(0) = 5$ and $y'(0) = -3$.

(b) Find the general solution to $9y'' - 12y' + 4y = 0$ and then find the solution with $y(1) = 4$ and $y'(1) = 7$.

(c) Find the general solution to $y'' + 9y = 0$, and then find the solution with $y(0) = 1$ and $y'(0) = 2$.

(d) Find the general solution to $y'' - 2y' + 5y = 0$, and then find the solution with $y(3) = 4$ and $y'(0) = 2$.

Problem 8. An object is on the end of a spring. Let x be the displacement of the object from its rest position. We assume Hooke's that the force from the spring on the object is proportional to the displacement. Let k be the constant of proportionality, which is called the spring constant. If the object has mass m and $x(t)$ is the displacement after time t, then Newton's second law ($F = ma$) becomves

$$m\frac{d^2x}{dt^2} = -kx$$

or letting $'$ be the derivative with respect to time

$$mx'' = -kx.$$

(a) Show that the general solution to this equation is

$$x(t) = c_1 \cos(\sqrt{k/m}\, t) + c_2 \sin(\sqrt{k/m}\, t).$$

(b) Recall that for a periodic function $c_1 \cos(\omega t) + c_2 \sin(\omega t)$ the period is $2\pi/\omega$. Assume the mass of the object is 1.4 and we measure the period of the motion and it is 4.3. Use this to compute the spring constant. □