Mathematics 242 Homework.

Problem 1. Let x satisfy the initial value problem

$$x'' + 5x' + 6x = 5 + 3e^{-2t} - 4\sin(2t) \qquad x(0) = 4, \quad x'(0) = -3$$

(a) Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$.
(b) Find the partial fraction decomposition of $X(s)$ and say what software or web site you used to do this. (there are partial fraction computers as Wolfram Alpha, Symbol Lab, and Voovers. The program Maple will do partial fractions instruction's here. I have been using SageMath, which can be down loaded for free here. Instructions for partial fractions in SageMath can be found here. If you know of anther program/website that you like to do partial fractions send me a link and I will pass the information along to the rest of the class.)
(c) Take the inverse Laplace transform and give the solution to the given initial value problem. □

Problem 2. This problem is just to show that the method works on higher order equations. Let $x(t)$ satisfy

$$x'''(t) - 8x'(t) = 4e^{2t} + 3e^{-2t}, \qquad x(0) = 1, \quad x'(0) = 2, \qquad x'''(0) = 3.$$

(a) Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$.
(b) Find the partial fraction decomposition of $X(s)$ and say what software or web site you used to do this.
(c) Take the inverse Laplace transform and give the solution to the given initial value problem. □

In class today we defined the ***convolution*** of the functions f and g as

$$f * g(t) = \int_0^t f(t-u)g(u)\,dx = \int_0^t f(u)g(it-u)\,du$$

and showed that

$$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Written in terms of the inverse transform this is

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \tag{1}$$

Here is an example of this in action. Let us solve

$$x''(t) - 3x'(t) + 2x(t) = 3H(t-5), \qquad x(0) = 1, \quad x'(0) = -2.$$

where $H(t)$ is the ***Heaviside function***

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t. \end{cases}$$

Letting $X(s) = \mathcal{L}\{x(t)\}$ and using the usual rules we find

$$s^2X - s + 2 - 3(sX - 1) + 2X = 3\mathcal{L}\{H(t-5)\}.$$

This can be rearranged as

$$(s^2 - 3s + 2)X = s - 5 + 3\mathcal{L}\{H(t-5)\}.$$

and therefore

$$X(s) = \frac{s-5}{s^2-3s+2} + \frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\}. \tag{2}$$

We have the partial fraction decompositions (SageMath again)

$$\frac{s-5}{s^2-3s+2} = \frac{4}{s-1} - \frac{3}{s-2}$$

so that

$$\mathcal{L}^{-1}\left(\frac{s-5}{s^2-3s+2}\right) = 4e^t - 3e^{2t}, \tag{3}$$

and

$$\frac{3}{s^2-3s+2} = \frac{3}{s-2} - \frac{3}{s-1}$$

and therefore

$$\mathcal{L}\{3e^{2t} - 3e^t\} = \frac{3}{s^2-3s+2}$$

and therefore

$$\frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\} = \mathcal{L}\{3e^{2t} - 3e^t\}\mathcal{L}\{H(t-5)\}$$

Therefore equation (1) above gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\}\right\} &= \left(3e^{2t} - 3e^t\right) * H(t-5) \\ &= \begin{cases} 0, & t < 5; \\ \displaystyle\int_5^t \left(3e^{2(t-u)} - 3e^{t-u}\right) du, & 5 \le t \end{cases} \\ &= \begin{cases} 0, & t < 5; \\ \dfrac{3(e^{2(t-5)} - 1)}{2} - 3(e^{t-5} - 1), & 5 \le t \end{cases}\end{aligned}$$

Putting this together with equations (2) and (3) gives us our final solution

$$x(t) = \begin{cases} 4e^t - 3e^{2t}, & t < 5; \\ 4e^t - 3e^{2t} + \dfrac{3(e^{2(t-5)} - 1)}{2} - 3(e^{t-5} - 1), & 5 \le t \end{cases}$$

Problem 3. Let $x(t)$ satisfy the initial value problem

$$x''(t) + 4x(t) = 3H(t-4), \qquad x(0) = 6, \quad x'(0) = 12$$

Use the method above to solve this equation. □

Problem 4. Find the inverse Laplace transforms of the following functions:

(a) $F(s) = \dfrac{3s+5}{(s-7)^2}$

(b) $G(s) = \dfrac{2s - 9}{2s^2 + 18}$

(c) $H(s) = \dfrac{3s + 12}{3s^2 + 12s + 21}$