Math 242 Test 2.

- *This is due on Friday, October 23 at midnight. You are to work alone in it. You can look up definitions and the statements of theorems we have covered in class. And if there is an integral where you want to use a computer, your calculator, or a source such as Wolfram Alpha to compute it, that is fine, but say that this is what you did. (For example "I computed $\int x^2 e^x\,dx$ using the program Maple".) Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg for direct help on the problems.*
- *Print your name on the first page of the exam.*
- *Use a dark pen or pencil so that the writing stands out even after being copied.*

Problem 1 (5 points). On a homework students were ask to find the general solution to

$$x^2y'' - 2xy' + 2y = x.$$

Alice gave the answer as

$$y = -x\ln(x) + C_1x + C_2x^2$$

while Bob gave the answer as

$$y = x - x\ln(x) + C_1x + C_2x^2.$$

Write a few sentences explaining why both answers are correct. □

Problem 2 (35 points). Solve the following initial value problems. Say what method you are using and show your work.

(a) $y'' - 3y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = -2.$
(b) $y'' - 3y' + 2y = \sin(x), \qquad y(0) = 2, \quad y'(0) = -1.$
(c) $y'' - 3y' + 2y = 2\cos(x) + 6\sin(x), \qquad y(0) = 5, \quad y'(0) = 11.$
(d) $y''+y = 0, \qquad y(\pi/2) = y_0, \quad y(\pi/2) = y_1$ where y_0 and y_1 are constants.
(e) $y'' + c^2y = \cos(ct), \qquad y(0) = 1, \quad y'(0) = 0$ where c is a constant. □

Problem 3 (10 points). Find a particular solution to $y''' + y' - 3y = 3\cosh(x)$. □

Problem 4 (10 points). Let y be a function that satisfies

$$y'' + 4y' + 2y = e^x \qquad y(0) = 3, \quad y'(0) = -4.$$

Find the Laplace transform of y. □

The Laplace transform can be used for systems of differential equations where there are more than one unknown functions. Here is an example. Consider functions $x(t)$ and $y(t)$ that satisfy

$$x'(t) = x(t) + y(t), \qquad x(0) = 3$$
$$y'(t) = x(t) - y(t), \qquad y(0) = -4$$

Taking $\mathcal{L}$ of these equation gives

$$\mathcal{L}\{x'(t)\} = \mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\}$$
$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\}$$

Then using our basic results about Laplace transforms we have

$$\mathcal{L}\{x(t)\} = s\mathcal{L}\{x(t)\} - x(0) = s\mathcal{L}\{x(t)\} - 3$$
$$\mathcal{L}\{y(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\} + 4$$

Plugging this into what we have already done gives

$$s\mathcal{L}\{x(t)\} - 3 = \mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\}$$
$$s\mathcal{L}\{y(t)\} + 4 = \mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\}$$

These can be rewritten as

$$\begin{aligned} (s-1)\mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} &= 3 \\ -\mathcal{L}\{x(t)\} + (s+1)\mathcal{L}\{y(t)\} &= -4 \end{aligned}$$

Doing some algebra (I used Cramer's rule and then checked my work using the computer program MathSage) we can solve for $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{y(t)\}$ to get

$$\begin{aligned} \mathcal{L}\{x(t)\} &= \frac{3s-1}{s^2-2} \\ \mathcal{L}\{y(t)\} &= \frac{-4s+7}{s^2-2}. \end{aligned}$$

Problem 5 (10 points). Let $x(t)$ and $y(t)$ satisfy

$$\begin{aligned} x'(t) &= 4x(t) + y(t), & x(0) &= 6 \\ y'(t) &= x(t) - 2y(t), & y(0) &= -12 \end{aligned}$$

Find the Laplace transforms $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{y(t)\}$. □

Problem 6 (15 points). (a) Show that $y_1 = x$ and $y_2 = x^{-1}$ are solutions to the differential equation

$$x^2y'' + xy' - y = 0.$$

(b) Use variation of parameters to find the general solution to

$$x^2y'' + xy' - y = -5 + \frac{6}{x}.$$ □

Problem 7 (15 points). For functions y_1 and y_2 we have defined the Wronskian to be

$$W = W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

(a) Show that the derivative of W is

$$W' = y_1y_2'' - y_1''y_2.$$

(b) Now assume that y_1 and y_2 are both solutions to the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

Show that the formula for the derivative W' can be simplified to

$$W' = -p(x)W$$

Hint: As y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$ it satisfies $y_1'' = -p(x)y_1' - q(x)y_1$, and likewise $y_2'' = -p(x)y_2' - q(x)y_2$. Use these in the formula for W' in part (a).

(c) Therefore W satisfies a first order linear equation. Set

$$P(x) = \int_0^x p(t)\, dt.$$

(That is $P(x)$ is the function with $P'(x) = p(x)$ and $P(0) = 0$, facts you can use in the rest of this problem.) Show

$$W = W(0)e^{-P(x)}$$

where $W(0) = y_1(0)y_2'(0) - y_1'(0)y_2(0)$ is the value W at $x = 0$. $\square$