Mathematics 546 Homework, October 25, 2020

Definition 1. A map $\varphi\colon G_1 \to G_2$ between groups is a ***homomorphism*** if and only if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in G_1$ $\square$.

That is φ is homomorphism if and only if it preserves products. We now show it preserves the identity elements, that has maps the identity of the first group to the identity of the second group.

Lemma 2. *Let G be a group and $b \in G$ satisfy $b^2 = b$. Then $b = e$.*

Problem 1. Prove this. $\square$

Proposition 3. *If $\varphi\colon G_1 \to G_2$ is a homomorphism between groups then*

$$\varphi(e_1) = e_2$$

where e_1 is the identity of G_1 and e_2 is the identity of G_2.

Problem 2. Prove this. *Hint:* As we did in class let $b = \varphi(e_1)$ and then show $b^2 = b$ and explain why this finishes the proof. $\square$

Proposition 4. *If $\varphi\colon G_1 \to G_2$ is a homomorphism between groups and $a \in G_1$, then*

$$\varphi(a^{-1}) = (\varphi(a))^{-1}.$$

Problem 3. Prove this. $\square$

Proposition 5. *If $\varphi\colon G_g \to G_2$ is a homomorphism and $a_1, a_2, \ldots, a_n \in G_1$, then*

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).$$

Proof. This is a standard induction argument. The base case is $n = 1$, that is $\varphi(a_1) = \varphi(a_1)$. (It might be more natural to use $n = 2$ and the base case as $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ is the definition of φ being a homomorphism.) To do the induction the induction hypothesis is

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).$$

then

$$\begin{aligned}
\varphi(a_1 a_2 \cdots a_n a_{n+1}) &= \varphi((a_1 a_2 \cdots a_n) a_{n+1}) \\
&= \varphi(\varphi(a_1 a_2 \cdots a_n)\varphi(a_{n+1}) && \text{(def. of homomorphism)} \\
&= \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)\varphi(a_{n+1}) && \text{(induction hypothesis)}
\end{aligned}$$

which closes the induction and completes the proof. $\square$

Proposition 6. *If $\varphi\colon G_g \to G_2$ is a homomorphism and $a \in G_1$ then*

$$\varphi(a^n) = (\varphi(a))^n$$

for all $a \in G_1$ and $n \in \mathbb{Z}$.

Proof. For $n \geq 1$ just let $a_1 = a_2 = \cdots a_n = a$ in Proposition 5 to get the result. For $n = 0$ we have $\varphi(a^0) = \varphi(e_1) = e_2 = (\varphi(a))^0$. Now assume true for $k \geq 0$. Then then for $n < 0$ we have $n = -k$ with $k > 0$ and thus

$$\varphi(a^n) = \varphi((a^{-1})^k) = (\varphi(a^{-1}))^k = ((\varphi(a))^{-1})^k = (\varphi(a))^{-k} = (\varphi(a))^n.$$

So we have covered all the cases, $n > 0$, $n = 0$, and $n < 0$. □

Recall that we have shown proven the following a few weeks ago:

Proposition 7. *If G is a group and $a \in G$ with $o(a) = n < \infty$. Then $a^k = e$, implies $n \mid k$.* □

Proposition 8. *If $\varphi\colon G_1 \to G_2$ is a homomorphism and $a \in G_1$ has $o(a) = n < \infty$. Then $o(\varphi(a)) \mid o(a)$. (That is the order of the image divides the order of original element.)*

Problem 4. Prove this. *Hint:* Reduced this to Proposition 7. □

Example 9. Let G_1 and G_2 be finite groups with $|G_1| = 17$ and $|G_2| = 42$. Let $\varphi\colon G_1 \to G_2$ be a homomorphism. Then $\varphi(a) = e_2$ for all $a \in G_1$. To see this note that a is an element of G_1 and therefore by Lagrange's theorem

$$o(a) \mid |G_1| = 17.$$

The image $\varphi(a)$ is an element of G_2 so (Lagrange again)

$$o(\varphi(a)) \mid |G_2| = 42.$$

By Proposition 8

$$o(\varphi(a)) \mid o(a) \mid 17.$$

Therefore $o(\varphi(a)) \mid 42$ and $o(\varphi(a)) \mid 17$. But $\gcd(17, 42) = 1$ and so the only positive integer that divides both 17 and 42 is 1. Whence $o(\varphi(a)) = 1$, which implies $\varphi(a) = e_2$. □

Proposition 10. *Let G_1 and G_2 be finite groups with $\gcd(|G_1|, |G_2|) = 1$ and $\varphi\colon G_1 \to G_2$ a homomorphism. Then $\varphi(a) = e_2$ for all $a \in G_1$.*

Problem 5. Prove this. □

Proposition 11. *Let G be a group and $g \in G$. Define a map $\varphi\colon G \to G$ by*

$$\varphi(x) = gxg^{-1}.$$

Then φ is a group homomorphism. □

Problem 6. Prove this. □

Proposition 12. *Let G be an Abelian group and let $k \in \mathbb{Z}$. Define $\varphi\colon G \to G$ by*

$$\varphi(a) = a^k$$

then φ is a homomorphism.

Problem 7. Prove this. □

Proposition 13. *Let G be a group and H a subgroup of G. Then the following are equivalent:*
(a) $gHg^{-1} = H$ *for all* $g \in G$.
(b) $gHg^{-1} \subseteq H$ *for all* $g \in G$.
(c) $gH = Hg$ *for all* $g \in G$.

Proof. We did this in class. □

Definition 14. A subgroup H of G is ***normal*** if and only if it ratifies the conditions of Proposition 13 □

Problem 8. Show that every subgroup of an Abalian group is normal. □

Recall that the center of the group G is

$$Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}.$$

That is $Z(G)$ is the set of elements of G that commute with all the elements of G.

Problem 9. Show the center of a group is a normal subgroup of G. □

We have seen that if H is a normal subgroup of G and we set

$$G/H = \{aH : a \in G\}$$

to be the set of cosets of H in G, then G/H with the group operation

$$(aH)(bH) = abH$$

is also a group, the ***quotient group of*** G ***by*** H.

Proposition 15. *Let H be a normal subgroup of G and define $\pi\colon G \to G/H$ by*

$$\pi(a) = aH.$$

Then π is a group homomorphism.

Problem 10. Prove this. □

Problem 11. Let $\varphi\colon G_1 \to G_2$ be a homomorphism between groups and let

$$K = \{a \in G_1 : \varphi(a) = e_2\}.$$

(a) Prove K is a subgroup of G_1.
(b) Show K is normal in G_1. (That is not only is K a subgroup, it is a normal subgroup.) □