Mathematics 554H Name:________________________________

Show your work to get credit.

1. (10 points) Let $S \subseteq \mathbb{R}$ be a nonempty subset of the real numbers, $\mathbb{R}$. (a) Define what it means for $b \in \mathbb{R}$ to be an ***upper bound*** for S.

(b) Define what it means for β to be a ***least upper bound*** for S (denoted $\beta = \sup(S)$).

(c) State the ***Least Upper Bound Axiom***.

(d) Use the Least Upper Bound Axiom to show the set $S = \{1.02, (1.02)^2, (1.02)^3, (1.02)^4, \ldots\}$ has no upper bound in $\mathbb{R}$.

2. (5 points) Let $f\colon X \to Y$ be a map between metric spaces and let $x_0 \in X$ and $y_0 \in Y$. Give the ε-δ definition of

$$\lim_{x \to x_0} f(x) = y_0.$$

3. (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = x^3.$$

Give an ε-δ proof that

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = 12.$$

4. (10 points) If $x, y \in \mathbb{R}$ with $|x-2| < \delta$ and $|y-3| < \delta$ where $0 < \delta < 1$ show

$$|xy - 6| < 6\delta.$$

5. (10 points) Let E be a metric space and $S \subseteq E$ a nonempty subset of E.

(a) Define what it means for S to be an ***open set***.

(b) Show that if U_1, U_2, and U_3 are open subsets of a metric space, then $U_1 \cap U_2 \cap U_3$ is also open.

(c) Give an example of open subsets $U_1, U_2, U_3, \dots$ of $\mathbb{R}$ such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

6. (10 points) (a) Define what it means for S to be a ***closed set***.

(b) Define what it means for p to be an ***adherent*** point of S.

(c) Show that if S is closed and p is an adherent point of S, then $p \in S$

7. (10 points) Let E be a metric space.

(a) Define what it means for E to be ***complete***.

(b) Show that if E is a complete metric space and F is a closed subset of E, then F is also complete.

8. (10 points) (a) Define E is ***connected***.

(b) Explain why the empty set $\varnothing$ is connected.

(c) Prove that if $E = A \cup B$ with each of A and B nonempty, connected and with $A \cap B \neq \varnothing$, then E is connected.

9. (10 points) (a) Define K is a ***compact*** subset of the metric space E.

(b) Show that if K_1 and K_2 are compact subsets of the metric space S that the metric space E that the union $K_1 \cup K_2$ is compact.

(c) Given an example of compact subsets $K_1, K_2, K_3, \ldots$ of $\mathbb{R}$ such that the union $\bigcup_{k=1}^{\infty} K_k$ is not compact.

(d) Show that any compact subset of a metric space can be covered with a finite number of balls of radius .001.

10. (10 points) (a) State the ***Intermediate Value Theorem*** for a continuous function $f\colon [a,b] \to \mathbb{R}$.

(b) Use that the continuous image of a connected set is connected and that the connected subsets of $\mathbb{R}$ are just the intervals to prove the form of the Intermediate Value Theorem just given.

(c) Use the Intermediate Value Theorem to prove the equation $\sqrt{x} = \dfrac{x^2}{1+x}$ has a positive solution. (You may assume the square root function is continuous.)

11. (5 free points) ***Have a good holiday break.***