Mathematics 554 Remarks on homework.

A problem on a recent homework that has caused some trouble is

Problem 1. Show that if $a_1, a_2, \ldots, a_n \in \mathbb{F}$ that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$$

with equality if and only if $a_1 = a_2 = \cdots a_n$.

The first problem some people was just what the problem was asking. It is really asking you to show two things:

(i) If if $a_1, a_2, \ldots, a_n \in \mathbb{F}$ then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$$

and

(ii) If equality holds, that is if

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 0,$$

then $a_1 = a_2 = \cdots = a_n = 0$.

This terminology is standard and you will see it again.

Proof of (i): We use induction. If $n = 1$, then this is just the statement that $a_1^2 \geq 0$. We have already proven this, but just to be complete, there are three cases, $a_1 > 0$, in which case $a_1^2 = a_1 a_1 > 0$ as the set of positive numbers. $a_1 = 0$, in which case $a_1^2 = 0 \geq 0$, and the last case is $a_1 < 0$, then $-a_1 > 0$ and $a_1^2 = (-a_1)(-a_1) > 0$ again as the set of positive numbers is closed under products.

For the induction step assume that the statement is true for n and let $a_1, a_2 \ldots, a_n, a_{n+1} \in \mathbb{F}$. Write

$$a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 = p + a_{n+1}^2$$

where $p = a_1^2 + a_2^2 + \cdots + a_n^2$. Then $p \geq 0$ by the induction hypothesis and $a_{n+1}^2 \geq 0$ by what we have just done. Therefore $a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 = p + a_{n+1}^2$ is the sum of two non-negative numbers and thus non-negative. This closes the induction. $\square$

Proof of (ii): Again we use induction. For $n = 1$ we have $a_1^2 = 0$. We have shown $ab = 0$ implies $a = 0$ or $b = 0$. When $a = b$ this gives that $a^2 = 0$ implies $a = 0$. This takes care of the $n = 1$ case.

Now assume it holds for n and assume, with the same notation as above

$$a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 = p + a_{n+1}^2 = 0.$$

From *(i)* we have $p \geq 0$ and $a_{n+1}^2 \geq 0$. If either $p > 0$ or $a_{n+1}^2 > 0$, then $p + a_{n+1}^2 > 0$, which is not the case, so $p = a_{n+1}^2 = 0$. Then by the induction hypothesis $a_1 = a_2 = \cdots = a_n = a_{n+1} = 0$, and we are done. $\square$