Mathematics 554 Homework.

We have defined open and closed subsets of a metric space and have a result that gives several different conditions for a set to be closed.

Theorem 1. *Let E be a metric space and $S \subseteq E$. Then the following are equivalent:*
(a) S is a closed subset of E.
(b) S contains all its adherent points.
(c) S contains the limits of all its convergent sequences, that is if $\langle p_n \langle_{n=1}^{\infty}$ is sequence from S which converges to some point p of E, then $p \in S$. $\square$

We want to give conditions on a sequence that implies it converges. Maybe our most basic result here is

Theorem 2. *A bounded monotone sequence in $\mathbb{R}$ converges.* $\square$

If we want convergent subsequence of a sequence in $\mathbb{R}$ we have

Theorem 3. *Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a bounded in $\mathbb{R}$. Then we have shown it has a monotone subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$. This subsequence is bounded and monotone and therefore convergent. $\square$

Definition 4. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then $\langle p_n \rangle_{n=1}^{\infty}$ is a ***Cauchy sequence*** if and only if for all $\varepsilon > 0$ there is a N such that $m, n \geq N$ implies $d(p_m, p_n) < \varepsilon$. $\square$

Being Cauchy is necessary for a sequence to converge:

Proposition 5. *Every convergent sequence is a Cauchy sequence.* $\square$

We have seen examples of Cauchy sequences that do not converge. One such example to let $E = (0, \infty)$ be the set of positive real numbers with the usual distance $d(x, y) = |x - y|$. Then the sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $x_n = 1/n$ is a Cauchy sequence in E, but $\langle x_n \rangle_{n=1}^{\infty}$ does not converge in E. This example feels a bit like a cheat as the sequence does converge in a larger space (that is in $\mathbb{R}$) and the fact that this Cauchy sequence does not converge somehow means that E should have a point (that is 0) which was "left out". That is in some sense the space E is not "complete".

Definition 6. Let E be a metric space. Then E is ***complete*** if and only if every Cauchy sequence in E converges to a point of E. $\square$

Our most important recent result is that what is our (or at least my) favorite metric space is complete.

Theorem 7. *The real numbers $\mathbb{R}$ with the metric $d(x, y) = |x - y|$ is a complete metric space.* $\square$

We were then able to use this to show

Theorem 8. *The metric space $\mathbb{R}^n$ with its usual metric is complete.* □

Once we have a complete space we get almost for free that some of its subsets are also complete. Recall that if E is a metric space with distance function $d(p, q)$ and $S \subseteq E$ is a nonempty subset of E then S is also a metric space by just restricting the distance function to S. That is for $p, q \in S$ the distance is still $d(p, q)$.

Theorem 9. *Let E be a complete metric space. Then a nonempty subset $S \subseteq E$ is complete if and only if S is a closed subset of E.*

Problem 1. Prove this along the following lines.

(a) First assume S is closed. We need to show that any Cauchy sequence $\langle p_n \rangle_{n=1}^{\infty}$ from S converges to a point of S. Use that E is complete to show there is a point $p \in E$ such that $\lim_{n\to\infty} p_n = p$. Now use Theorem 1 to show $p \in S$.

(b) To prove the converse assume that S is complete. To show S is closed use Theorem 1: to show S is closed we just need to show that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S which converges to a point, p, of E that $p \in S$. Since the sequence converges in E, it is a Cauchy sequence. Now explain why S being complete implies $p \in S$. □

Definition 10. A metric space E is ***sequentially compact*** if and only if every sequence $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence which converges to a point of E. □

One of our most recent results is

Theorem 11 (BolzanoWeierstrass theorem)**.** *Every closed bounded subset of $\mathbb{R}$ is sequentially compact.*

Problem 2. Prove this. *Hint:* Let F be a closed bounded subsequence of $\mathbb{R}$ and $\langle x_n \rangle_{n=1}^{\infty}$ a sequence of points from F. Then use Theorem 3 to get a convergent (in $\mathbb{R}$) subsequence and then use Theorem 1 to show the limit of this subsequence is in fact in F. □

This was generalize to higher dimensions.

Theorem 12 (General Bolzano-Weierstrass Theorem)**.** *Any closed bounded set of $\mathbb{R}^n$ is sequentially compact.* □

This has a converse.

Theorem 13. *Let S be a subset of $\mathbb{R}^n$ that is sequentially compact. Then S is closed and bounded. (By **bounded** in this setting we mean there is a constant M such that $\|p\| \leq M$ for all $p \in S$.)*

Problem 3. Prove this. *Hint:*

(a) First show S is bounded. One way is to assume, towards a contradiction, that S is not bounded. Then for each positive integer m there is a point $p_m \in S$ with $\|p_m\| \geq m$. Show that no subsequence of $\langle p_m \rangle_{m=1}^{\infty}$

is Cauchy and therefore the sequence has no convergent subsequence. Explain why this is a contradiction.

(b) Show that S is bounded. The same circle of ideas that used Problem 1 (b) should work here. $\square$