Mathematics 554 Solutions to some homework problems.

Problem 2. In this problem we make sense of the expression

$$\alpha = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}$$

and find its value. To start let

$$x_0 = \sqrt{2}$$

and define x_1, x_2, x_3, ... recursively by

$$\begin{aligned} x_1 &= \sqrt{2 + x_0} \\ x_2 &= \sqrt{2 + x_1} \\ x_3 &= \sqrt{2 + x_2} \\ &\vdots \\ x_{n+1} &= \sqrt{2 + x_n} \end{aligned}$$

(a) Give the explicit formulas for x_1, x_2 and x_3. Our goal now is to show that $\lim_{n\to\infty} x_n$ exists. This will then be our definition of α.

(b) Note that $x_1 = \sqrt{2 + x_0} > \sqrt{2} = x_0$. Therefore $x_1 > x_0$. Use this as a base case of an induction to show that $\langle x_n \rangle_{n=1}^{\infty}$ is an increasing sequence. *Hint:* There are many ways to do this. One way is to use the rationalizing the numerator trick that was worked for us before. To be a little more explicit show

$$x_{n+1} - x_n = \sqrt{2 + x_n} - \sqrt{2 + x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{2 + x_n} + \sqrt{2 + x_{n-1}}}$$

and use this to do the induction step.

(c) Use induction to show $x_n < 2$ for all n.

(d) Therefore $\langle x_n \rangle_{n=1}^{\infty}$ is a bounded monotone sequence and thus is convergent. Let

$$\alpha = \lim_{n\to\infty} x_n$$

So we have made sense of what α should mean, but we still need to find its value. Justify that

$$\lim_{n\to\infty} x_{n+1} = \alpha$$

and

$$\lim_{n\to\infty} \sqrt{2 + x_n} = \alpha$$

and therefore we can take limits in

$$x_{n+1} = \sqrt{2 + x_n}$$

to get

$$\alpha = \sqrt{2 + \alpha}.$$

Solve this to find α. □

Solution. (a)

$$x_1 = \sqrt{2+\sqrt{2}}$$
$$x_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$
$$x_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

(b) The base case is done. The induction hypothesis is $x_n > x_{n-1}$, that is $x_n - x_{n+1}$. A calculation I leave to you now gives

$$x_{n+1} - x_n = \sqrt{2+x_n} - \sqrt{2+x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{2+x_n}+\sqrt{2+x_{n-1}}} > 0$$

by the induction hypothesis. This closes the induction.

(c) Again the base case is done. So assume $x_n < 2$, then

$$x_{n+1} = \sqrt{2+x_n} < \sqrt{2+2} = 2.$$

(d) Now that we know that $x_1, x_2, x_2, x_3, \ldots$ is a bounded increasing sequence, we have that this sequence converges. Let

$$\alpha = \lim_{n\to\infty} x_n$$

Some of the results we have proven recently given imply

$$\lim_{n\to\infty} x_{n+1} = \alpha, \qquad \lim_{n\to\infty} \sqrt{2+x_n} = \sqrt{2+\alpha}.$$

Therefore taking the limit as $n \to \infty$ in

$$x_{n+1} = \sqrt{2+x_n}$$

gives

$$\alpha = \sqrt{2+\alpha}.$$

Square this

$$\alpha^2 = 2 + \alpha$$

that is

$$\alpha^2 - \alpha - 2$$

and we can use the quadratic formula to get

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2} = 2, -1.$$

As α is positive this gives $\alpha = 2$. □

Problem 3. (Extra credit) Let $b > 0$ can your do a similar analysis to make sense of and find the value of

$$\beta = \sqrt{b + \sqrt{b + \sqrt{b + \sqrt{b + \sqrt{b + \cdots}}}}}$$

□

Solution. We start by guessing the value of β. If it exists it should satisfy

$$\beta = \sqrt{b + \beta}.$$

Square this and rearrange a bit to get

$$\beta^2 - \beta - b = 0$$

so (quadratic formula)

$$\beta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-b)}}{2} = \frac{1 \pm \sqrt{4b + 1}}{2}.$$

If β exists, then it is positive and thus our conjectured value of β is

$$\beta = \frac{1 + \sqrt{4b + 1}}{2}$$

Note

$$x_0 = \sqrt{b} < \frac{1 + \sqrt{4b + 1}}{2} = \beta$$

Assume $x_n < \beta$, then

$$x_{n+1} = \sqrt{b + x_n} < \sqrt{b + \beta} = \beta$$

because $sqrt b + \beta = \beta$ (this is the equation we solved to define β.) Thus induction implies the $x_0, x_1, x_2, \ldots$ is is is bounded above by β.

We also have $x_1 = \sqrt{b + x_0} > \sqrt{b} = x_0$. Assume $x_n > x_{n-1}$ then (I am leaving some of the algebra to you)

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{\sqrt{b + x_n} + \sqrt{b + x_{n-1}}} > 0$$

and yet anther induction implies $x_1, x_2, x_3, x_4, \ldots$ is an increasing sequence. Thus it is bounded and monotone and therefore $\lim_{n\to\infty} x_n = L$ exists. Taking the limit in

$$x_{n+1} = \sqrt{b + x_n}$$

gives that L satisfies

$$L = \sqrt{b + L}$$

but we have see that the only solution to this is β and therefore

$$L = \beta = \frac{1 + \sqrt{4b + 1}}{2}.$$

□

Problem 4. Here is an easier variant on the theme of the last couple of problems. Let $x_0 = 0$ and define a sequence by

$$x_{n+1} = \frac{2x_n}{3} + 42.$$

Show that this sequence is increasing and bounded above and find its limit. *Hint:* The increasing part should not be too hard. To get an upper bound just try some large number. If nothing else, 666 works.

Solution. First $x_0 = 0 < 666$ and if $x_n < 666$, then

$$x_{n+1} = \frac{2}{3}x_n + 42 < \frac{2}{3}666 + 42 = 486 < 666$$

and thus induction gives that the sequence is bounded above.

Also $x_1 = 42 > 0 = x_0$. Assume $x_n > x_{n-1}$, then

$$x_{n+1} = \frac{2}{3}x_n + 42 > \frac{2}{3}x_{n-1} + 42 = x_n$$

and so induction implies $\langle x_n \rangle_{n=0}^{\infty}$ is increasing. Therefore this sequence is bounded and increasing and therefore it is convergent. Let $L = \lim_{n\to\infty} x_n$. Then taking the limit in

$$x_{n+1} = \frac{2}{3}x_n + 42$$

yields

$$L = \frac{2}{3}L + 42.$$

Solving this for L gives $L = 126$ and therefore

$$\lim_{n\to\infty} x_n = L = 126.$$

□