Mathematics 554 Homework.

Here are some problem to get familiar with open covers and compact sets.

Definition 1. Let E be a metric space and S a subset of E. Then $\mathcal{U}$ is an ***open cover*** of S if and only if the following two conditions holds.

(a) Each element of $\mathcal{U}$ is a open subset of E.
(b) For each $a \in S$ there is a $U \in \mathcal{U}$ with $a \in U$. □

The second of these conditions can be rewritten as

$$S \subseteq \bigcup_{U \in \mathcal{U}} U.$$

Definition 2. Let E be a metric space and $S \subseteq E$. Then S is ***compact*** if and only if every open cover of S has a finite subset that is also an open cover of S.

To be a little more explicit, if S is compact and $\mathcal{U}$ is an open cover of S then there is a finite subset $\mathcal{U}_0 = \{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$ such that $S \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

Until we have some theorems about compact sets upper our belt, the way we will use compactness is to make a smart choice of an open cover and use that it has a finite subcover to get our result. Here are some examples.

Proposition 3. *Let S be a compact subset of the metric E and p_0 any point of E. Then there is a $r > 0$ so that $S \subseteq B(p_0, r)$. (To use terminology we have used before, this says that any compact subset of a metric space is bounded.)*

Problem 1. Prove this. *Hint:*

(a) Show that $\mathcal{U} = \{B(p_0, r) : r > 0\}$ (that is the collection of all open balls centered at p_0) is an open cover of S.
(b) Let $\mathcal{U}_0 = \{B(p_0, r_1), B(p_0, r_2), \ldots, B(p_0, r_n)\}$ be a finite subset of $\mathcal{U}$ that covers S and show that $S \subseteq B(p_0, r)$ where $r = \max\{r_1, r_2, \ldots, r_n\}$. □

Proposition 4. *If S is a compact subset of the metric space E, then S contains all its adherent points and therefore is closed.*

Problem 2. Prove this. *Hint:* Toward a contradiction assume that S is compact, and that S has an adherent point p with $p \notin S$.

(a) For each $r > 0$ let $U_r = \mathcal{C}(\overline{B}(p, r)) = \{q \in E : d(p, q) > r\}$. In English U_r is the set of points of E that are at a distance greater than r from p. This is an open set (as it is the compliment of the closed ball $\overline{B}(p, r)$ and you do not need to prove this). Show $\mathcal{U} = \{U_r : r > 0\}$ is an open cover of S.
(b) Let $\mathcal{U}_0 = \{U_{r_1}, U_{r_2}, \ldots, U_{r_n}\}$ be a finite subset of E that covers S, let $r = \min\{U_{r_1}, U_{r_2}, \ldots, U_{r_n}\}$, and show $B(p, r) \cap S = \varnothing$. Explain why this contradicts that p is an adherent point of S. □

Proposition 5. *Let S be a compact subset of the metric space E and let F be a closed set of E with $F \subseteq S$. Then F is also compact. (That is closed subsets of compact sets are compact.)*

Problem 3. Prove this. *Hint:* Let $\mathcal{U}$ be an open cover of F. Show $\mathcal{V} = \{\mathcal{C}(F)\} \cup \mathcal{U}$ is an open over of S and use this to show that $\mathcal{U}$ has a finite subset that covers F. □

Problem 4. Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in E, say $\lim_{n\to\infty} p_n = p$. Let

$$S = \{p\} \cup \{p_n : n = 1, 2, 3, \ldots\}.$$

Show S is compact. □

Problem 5. Problem 3.57 on page 68 of *Notes on Analysis.* □