Mathematics 554: notes on disconnections.

Here we say a bit more about showing that sets are disconnected.

To start we say a little more about metric space. Recall that if X is a metric space with distance function d and $E \subseteq X$ is a subset of X, then E is a metric space by just using the distance function d restricted to pairs of points in E. With notation that I hope is self-explanatory if $p, q \in E$ we have

$$d_E(p,q) = d_X(p,q).$$

Then an open ball $B_E(p,r)$ in E is

$$B_E(p,r) = \{x : x \in E \text{ and } d(p,x) < r\}.$$

The definition of the intersection of two sets is

$$A \cap B = \{xx \in A \text{ and } x \in B\}$$

and the definition of an open ball in the larger space X is

$$B(p,r) = \{x : d(p,x) < r\}$$

and so

$$\begin{aligned} B_E(p,r) &= \{x : x \in E \text{ and } d(p,x) < r\} \\ &= E \cap \{x : d(p,x) < r\} \\ &= E \cap B(p,r). \end{aligned}$$

Proposition 1. *Let E be a subset of the metric space X and let U be an open subset of X. Then the intersection $U \cap E$ is an open subset of E.*

Proof. To show that $U \cap E$ is open we need to show that for each $p \in U \cap E$ there is an $r > 0$ so that $B_E(p,r) \subseteq U \cap E$. If $p \in U \cap E$, then $p \in U$ and U is open in X and therefore there is $r > 0$ such that $B(p,r) \subseteq U$. But then

$$B_E(p,r) = E \cap B(p,r) \subseteq E \cap U$$

which finishes the proof. □

Definition 2. Let A and B be subsets of a metric space X. Then the open sets U and V ***separate*** A and B if and only if $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.

Theorem 3. *Let A and B be non-empty subsets of a metric space X which are separated by two open sets U and B and let $E = A \cup B$. Then E is disconnected.*

Proof. We show $E = A \cup B$ is a disconnection of E. We are given that A and B are non-empty. As U and V separate A and B we also have $U \cap V = \varnothing$. Thus

$$A \cap B \subseteq U \cap V = \varnothing.$$

Finally note

$$\begin{aligned} U \cap E &= U \cap (A \cup B) \\ &= (U \cap A) \cup (U \cap B) \\ &= A \cup \varnothing \\ &= A. \end{aligned}$$

where we have used $U \cap A = A$ as $A \subseteq U$ and $A \cap V \subseteq U \cap V = \varnothing$ and thus $A \cap V = \varnothing$. Therefore A is the intersection of an open set of the larger space X with E which by Proposition 1 implies A is open in E. A similar argument shows that B is open in E. Thus $E = A \cup B$ is a disconnection of E as claimed. □