Mathematics 554 Homework.

Since our test is on Monday, I will not collect this in the usual way, but you will have a quiz on it on Friday. So look at it before Wednesday and we can go over anything you do not understand.

Proposition 1. *Let E be a metric space and U and V disjoint open subsets of E. Show that if S is a connected subset of E and $S \subseteq (U \cup V) \neq \varnothing$, then either $S \subseteq U$ or $S \subseteq V$.*

Problem 1. Prove this. $\square$

Proposition 2. *Let E be a metric space and A a clopen subset of E. Then for an connected subset S of E either $S \subseteq A$ or $S \subseteq \mathcal{C}(A)$.*

Problem 2. Prove this. *Hint:* If A is clopen, then $U = A$ and $V = \mathcal{C}(A)$ are open disjoint subsets of E. $\square$

Theorem 3. *Let E be a metric space and $\{S_\alpha : \alpha \in I\}$ a collection of connected subsets of E. Assume there is a $\beta \in I$ such that*

$$S_\alpha \cap S_\beta \neq \varnothing \qquad \text{for all } \alpha \in I.$$

Then the union

$$S = \bigcup_{\alpha \in I} S_\alpha$$

is connected.

Problem 3. Prove this. *Hint:* Here as an argument that is a little different than the one we did in class. By replacing E with S we can assume $E = S$. Assume that A is a clopen subset of $E = S$ and we want to show that $A = \varnothing$ or $S = E$. Use Proposition 2 to show that for each α that $S_\alpha \subseteq A$ or $S_\alpha \subseteq \mathcal{C}(A)$. First assume $S_\beta \subseteq A$ and $S_\alpha \subseteq A$ for all α and therefore $E = \bigcup_{\alpha \in I} S_\alpha \subseteq A$ and therefore $A = E$. Do a similar argument in the case $S_\beta \subseteq \mathcal{C}(A)$.

Theorem 4. *Let $f \colon X \to Y$ be a Lipschitz map between metric spaces and let $U \subseteq Y$ be an open subset of Y. Then*

$$U = \{x \in X : f(x) \in V\}$$

is an open subset of X.

Problem 4. Prove this. *Hint:* Because f is Lipschitz there is a constant $M > 0$ such that for all $x_1, x_2 \in X$ the inequality

$$d_Y(f(x_1), f(x_2)) \leq M d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

(a) Let $x_0 \in U$, then by definition $f(x_0) \in V$. Explain why there is $r > 0$ such that $B_Y(f(x_0), r) \subseteq V$. (Do not make this hard, the answer should just be "This follows form the definition of V being an <u>*fill in blank*</u> set."

(b) Use the Lipschitz condition to show if $x \in B_X(x_0, r/M)$, then $f(x) \in B_Y(f(x_0), r) \subseteq V$.
(c) Explain why (b) implies U is open. (Again to not make this hard.)

Proposition 5. *Let X be a metric space and $f\colon X \to \mathbb{R}$ a Lipschitz map. Then the image*

$$f[X] = \{f(x) : x \in [a, b]\}$$

is connected.

Problem 5. Prove this. *Hint:* Toward a contradiction assume that $f[X]$ is not connected and let $f[X] = A \cup B$ be a disconnection of X. Then use Theorem 4 to show

$$X = \{x \in X : x \in A\} \cup \{x \in X : x \in B\}$$

is a disconnection of X. □

Definition 6. Let $f\colon X \to Y$ be a map between metric spaces. Then f is ***continuous*** at $x_0 \in X$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $x \in X$

$$d_X(x, x_0) < \delta \qquad \text{implies} \qquad d_Y(f(x), f(x_0)) < \varepsilon.$$

Problem 6. Memorize this. □