Mathematics 554 Homework.

Definition 1. Let $f\colon X \to Y$ be a function between sets. Then a function $g\colon Y \to X$ is the ***inverse*** of f if and only if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. $\square$

Proposition 2. *If $f\colon X \to Y$ has a inverse, then it is bijective, that is it is both injective (one to one) and surjective (onto).*

Problem 1. Prove this. *Hint:* Let g be an inverse of f. To show f is injective we need to show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To do this just apply g to both sides of $f(x_1) = f(x_2)$ and use that $g(f(x)) = x$.

To show f is surjective we need to show that for all $y \in Y$ there is $x \in X$ with $f(x) = y$. Use the definition of inverse inverse to show $f(g(y)) = y$. $\square$

Proposition 3. *If $f\colon X \to Y$ has in inverse, it is unique.*

Problem 2. Let $g_1, g_2\colon Y \to X$ be inverses of f. We wish to show $g_1 = g_2$. Let $y \in Y$. Then $f(g_2(y)) = y$ and thus

$$g_1(y) = g_1(f(g_2(y))).$$

Now explain why $g_1(f(g_2(y))) = g_2(y)$. $\square$

The converse of Proposition 2 is true.

Theorem 4. *A function $f\colon X \to Y$ has an inverse if and only if it is bijective.*

Problem 3. Prove this. *Hint:* In light of Proposition 2 you only need show that a bijective function has an inverse. Let $f\colon X \to Y$ be bijective. That it is surjective and therefore for each $y \in Y$ then is an $x \in X$ with $f(x) = y$. Use that f is injective to explain why this x is unique. Therefore if we let $g(x) = y$ for the unique x with $f(x) = y$, this defines a function $g\colon Y \to X$. Show this is the inverse of f. $\square$

The inverse of f, when it exists, is denoted f^{-1} (and you have to glen out from the context if f^{-1} is the inverse function or the reciprocal $1/f$).

Problem 4. Let $f\colon X \to Y$ be a bijective function. Show the inverse $f^{-1} : Y \to X$ is also bijective and that $(f^{-1})^{-1} = f$.

Problem 5. Let $f\colon X \to Y$ be a bijective function and let $S \subseteq X$. Show that the preimage of S by f^{-1} is just the image of S by f. That is

$$(f^{-1})^{-1}[S] = f[S].$$

$\square$

Theorem 5. *Let $f\colon X \to Y$ be a continuous bijective function between the metric spaces X and Y with X compact. Then the inverse $f^{-1}\colon Y \to X$ is also continuous.*

Problem 6. Prove this along the following lines:

(a) Show that Y is compact. *Hint:* Continuous images of compact sets are compact.

(b) To show f^{-1} is continuous it is enough to show that preimages by f^{-1} of closed sets are closed. By Problem 5 this just means that we need to show for each closed subset $C \subseteq X$ that

$$(f^{-1})^{-1}[S] = f[S]$$

is closed. (Nothing for you to do here other than copy this.)

(c) Now put together the facts that closed subsets of compact sets are compact, that compact subsets of a metric space are closed, and that continuous images of compact sets are compact to finish the proof. $\square$