Mathematics 554 Homework.

This homework is to correct some

Definition 1. Let $f\colon X \to Y$ be a function between metric space, $x_0 \in X$ and $y_0 \in Y$

$$\lim_{x \to x_0} f(x) = y_0$$

if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < d_X(x, x_0) < \delta \qquad \text{implies} \qquad d_Y(f(x), y_0)) < \varepsilon. \qquad \square$$

Proposition 2. *Let X and Y be metric spaces, $f\colon X \to Y$ any function, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\lim_{x \to x_0} f(x) = y_0$$

Proof. Let $\varepsilon > 0$ and let

$$\delta = \frac{d_X(x, x_0)\varepsilon}{1 + d_Y(f(x), y_0)}$$

Then if $0 < d_X(x, x_0) < \delta$

$$\begin{aligned}
d_Y(f(x), y_0) &= \frac{d_Y(f(x), y_0)}{d_X(x, x_0)}\, d_X(x, x_0) \\
&\leq \frac{d_Y(f(x), y_0)}{d_X(x, x_0)}\, \delta \\
&= \frac{d_Y(f(x), y_0)}{d_X(x, x_0)} \left(\frac{d_X(x, x_0)\varepsilon}{1 + d_Y(f(x), y_0)} \right) \\
&= \frac{d_Y(f(x), y_0)\varepsilon}{1 + d_Y(f(x), y_0)} \\
&< \varepsilon.
\end{aligned}$$

This shows if $0 < d_X(x, x_0) < \delta$, then $d_Y(f(x), y_0) < \varepsilon$ which verifies the definition of $\lim_{x \to x_0} f(x) = y_0$. $\square$

If we let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + x$ then by varying y_0 we get all of the following limits:

$$\begin{aligned}
\lim_{x \to 5} (x^2 + x) &= 30 \\
\lim_{x \to 5} (x^2 + x) &= 17 \\
\lim_{x \to 5} (x^2 + x) &= 42
\end{aligned}$$

which would contradict uniqueness of limits. So Proposition 2 must be false.

Problem. Find the mistake in the proof of Proposition 2. *Hint:* This has a one or two sentence answer and does not involve any bad algebra or trickiness in the calculation. $\square$