Mathematics 554 Homework.

Definition 1. Let $f\colon [a,b] \to \mathbb{R}$ be a function. Then f is ***Lipschitz*** if and only if there is a constant $K > 0$ such that the inequality

$$|f(x_2) - f(x_1)| \le K|x_2 - x_1|$$

holds for all $x_1, x_2 \in [a,b]$. □

Here is an example of using this definition and also some practice with inequalities.

Proposition 2. *If f is Lipschitz on $[a,b]$ and for some $c \in [a,b]$ we have $f(c) > 0$, the for any x with $|x - c| < |f(c)|/K$ we also have*

$$f(x) > 0.$$

Proof. This is just stringing together the definition and several of our tricks. Assume $|x - c| \le |f(c)|/K$, then

$$\begin{aligned} f(x) &= f(c) + (f(x) - f(c)) && \text{(adding and subtracting trick)} \\ &\ge f(c) - |f(x) - f(c)| && \text{(as } (f(x) - f(c)) \ge |f(x) - f(c)|\text{)} \\ &\ge f(c) - K|x - c| && \text{(as } |f(x) - f(c)| \le K|x - c|\text{)} \\ &> f(c) - K\frac{f(c)}{K} && \left(\text{as } |x - c| \le \frac{|f(c)|}{K}\right) \\ &= 0. \end{aligned}$$

□

Problem 1. If f is Lipschitz on $[a,b]$ and for some $c \in [a,b]$ we have $f(c) < 0$, show that for any x with $|x - c| < |f(c)|/K$ we also have

$$f(x) < 0.$$

□

Problem 2. This generalizes what we have just done. Again let $f\colon [a,b] \to \mathbb{R}$ be Lipschitz. Show that for any $x_0 \in [a,b]$ and any $\varepsilon > 0$ the for any $\in [a,b]$ with

$$|x - x_0| < \frac{\varepsilon}{K}$$

the inequalities

$$|f(x) - f(x_0)| < \varepsilon$$

and

$$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon.$$

Problem 3. Show the function $f(x) = x^2$ is Lipschitz on an interval $[0,b]$. (The Lipschitz K will depend on the number b.)

Problem 4. Do Problem 2.45 on Page 42 of *Notes on Analysis.* □