Mathematics 554 Homework.

Some review of set theory. Let E be a set which we think of as the "universal set", that is every other set we are considering is a subset of this set. I am assuming you know what is means for x to be an element of a set A, written $x \in A$. If A and B are subsets of E, then A is a ***subset*** of B (written as $A \subseteq B$ if and only if each element of A is also an element if B.

$$A \subseteq B \quad \iff \quad (x \in A \implies x \in B).$$

Two sets are equal if and only if they have the same elements:

$$A = B \quad \iff \quad (x \in A \iff x \in B).$$

If $P(x)$ is a statement about elements of E, for example "x is an even integer" then the set of elements of E where x is true written as

$$\{x : P(x)\}.$$

In some textbooks this is called ***set builder notation*** For example if the universal set is the set of real numbers, than

$$\{x : x^2 = 4\} = \{2, -2\}.$$

The set with no elements is the ***empty set***, that is and is denoted by $\varnothing$:

$$\varnothing = \{\}.$$

If S_i with $i \in I$ is a collection of subsets of E, that is for each $i \in I$ we have $S_i \subseteq E$, then the ***union*** of these sets is

$$\bigcup_{i \in I} S_i = \{x : x \in S_i \text{ for at least one } i \in I\}.$$

When there are only a finite number of sets $S_1, S_2, \ldots, S_n$ this is often written as

$$\bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n.$$

Problem 1. Use Archimedes' Axiom to show that if $E = \mathbb{R}$, then

$$\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}.$$

□

We also have the ***intersection*** of our collection of sets:

$$\bigcap_{i \in I} S_i = \{x : x \in S_i \text{ for all } i \in I\}.$$

Again in the case of a finite number of sets we can write

$$\bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n.$$

Problem 2. Use Archimedes' Axiom to show that if $E = \mathbb{R}$, then

$$\bigcap_{n=1}^{\infty} (-1/n, 1 + 1/n) = [0, 1].$$

(This gives us anther example of an infinite collection of open sets whose intersection is not open.) □

Proposition 1. *It* $A \subseteq E$ *then*

$$A \cap E = A, \quad A \cup E = E, \qquad A \cap \varnothing = \varnothing, \quad A \cap E = A.$$

Proof. A very basic definition chase through the definitions. □

There are distributive laws for union over intersection and intersection over union.

Proposition 2. *Let* S_1 *with* $i \in I$ *be a collection of subsets of* E *and* A *any subset of* E*. Then*

$$A \cap \bigcup_{i \in I} S_i = \bigcup_{i \in I} (A \cap S_i) \tag{1}$$

$$A \cup \bigcap_{i \in I} S_i = \bigcap_{i \in I} (A \cup S_i) \tag{2}$$

For finite unions and intersections these become

$$A \cap (S_1 \cup S_2 \cup \cdots \cup S_n) = (A \cap S_1) \cup (A \cap S_2) \cup \cdots \cup (A \cap S_n),$$
$$A \cup (S_1 \cap S_2 \cap \cdots \cap S_n) = (A \cup S_1) \cap (A \cup S_2) \cap \cdots \cap (A \cup S_n).$$

Proof of (1). This is a definition chase:

$$\begin{aligned} A \cap \bigcup_{i \in I} S_i &= \left\{ x : x \in A \text{ and } x \in \bigcup_{i \in I} S_i \right\} \\ &= \{x : x \in A \text{ and (for some } i \in I \text{ we have } x \in S_i)\} \\ &= \{x : \text{for some } i \in I \text{ we have } x \in A \text{ and } x \in S_i\} \\ &= \{x : \text{for some } i \in I \text{ we have } x \in A \cap S_i\} \\ &= \bigcup_{i \in I} (A \cap S_i). \end{aligned}$$

□

Problem 3. Prove Equation (2) holds. □

Proposition 3. *If* $A \subseteq S_i$ *for all* $i \in I$*, then*

$$A \subseteq \bigcap_{i \in I} S_i$$

Proof. Hopefully this is pretty much obvious from the definitions, here is are the details. Let $a \in A$, then for each $i \in I$ we have $x \in S_i$ as $A \subseteq S_i$. Therefore $x \in S_i$ for all i and when $x \in \bigcap_{i \in I} S_i$. This holds for all $x \in A$ and therefore $A \subseteq \bigcap_{i \in I} S_i$. □

Proposition 4. *If* $S_i \subseteq A$ *for all* $i \in I$*, then*

$$\bigcap_{i \in I} S_i \subseteq A.$$

Problem 4*.* Prove this. □

Problem 5*.* Here is a set up that will come up several times during the term. Let $A_1 \subseteq A_2 \subseteq \cdots A_n$. Then show

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_1$$
$$A_1 \cup A_2 \cup \cdots \cup A_n = A_n.$$

One final operation is the compliment of a subset. If $A \subseteq E$, then the ***compliment*** of A is the set of points of E not in A:

$$\mathcal{C}(A) := \{x : x \notin A\}.$$

Taking compliments of unions and intersections turns out to be easy.

Proposition 5 (De Morgan's laws)**.** *Let* S_i *with* $i \in I$ *be a collection of subsets of* E*. Then*

$$\mathcal{C}\Big(\bigcup_{i \in I} S_i\Big) = \bigcap_{i \in I} \mathcal{C}(S_i)$$
$$\mathcal{C}\Big(\bigcap_{i \in I} S_i\Big) = \bigcup_{i \in I} \mathcal{C}(S_i)$$

Problem 6*.* Pick one of these two and prove it. □