Mathematics 554 Homework.

Problem 2.48. Show for any real number $a > 0$ that the inequality

$$a + \frac{1}{a} \geq 2$$

holds with equality if and only if $a = 1$.

Solution. As $a > 0$ we know (Lipschitz Intermediate value Theorem) that a has a positive square root $\sqrt{a}$. Then a bit of algebra shows

$$a + \frac{1}{a} = 2 + \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 2,$$

where the inequality holds because squares are non-negative. If equality holds then the squared term must vanish and thus

$$\sqrt{a} - \frac{1}{\sqrt{a}} = 0.$$

Multiply through by $\sqrt{a}$ to get

$$a - 1 = 0$$

and therefore $a = 1$. That equality holds when $a = 1$ is easy to check. □

Problem 2.50.

Solution. We did this in class. □

Problems 3.1 and 3.2. Let E be a nonempty subset of $\mathbb{R}^n$ and define the distance between points of E by

$$d(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\|.$$

Show this makes E into a metric space. (Note when $n = 1$ and for $p \in \mathbb{R}$, that is p is just a real number, we have $\|p\| = \sqrt{p^2} = |a|$ so Problem 3.2 implies problem 3.1.)

Solution. We need to check the four defining properties of a metric.

(a) $d(\vec{p}, \vec{q}) \geq 0$. (Non-negativity of distance function.)
This as norms of vectors are non-negative $d(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\| \geq 0$.

(b) $d(\vec{p}, \vec{q}) = 0$ implies $\vec{p} = \vec{q}$.
The only vector with norm zero is $\vec{0}$, therefore $d(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\| = 0$ implies $\vec{p} - \vec{q} = 0$, that is $\vec{p} = \vec{q}$.

(c) $d(\vec{p}, \vec{q}) = d(\vec{q}, \vec{p})$ (symmetry of the distance function.)
The norm of a vector and its negative are the same: $\| - \vec{v}\| = \|\vec{v}\|$. Applying this to $\vec{v} = \vec{p} - \vec{q}$ gives

$$d(\vec{q}, \vec{p}) = \|\vec{q} - \vec{p}\| = \| - (\vec{p} - \vec{q})\| = \|\vec{p} - \vec{q}\| = d(\vec{p}, \vec{q}).$$

(d) $d(\vec{p},\vec{q}) \leq d(\vec{p},\vec{q}) + d(\vec{q},\vec{r})$ (the triangle inequality.)

This follows from the triangle inequality $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ for vectors and the adding and subtracting trick.

$$\begin{aligned} d(\vec{p},\vec{q}) &= \|\vec{p}-\vec{r}\| \\ &= \|\vec{p}-\vec{q}+\vec{q}-\vec{r}\| \\ &\leq \|\vec{p}-\vec{q}\| + \|\vec{q}-\vec{r}\| \\ &= d(\vec{p},\vec{q}) + d(\vec{q},\vec{r}). \end{aligned}$$

□

Problem 3.3. Let E be a metric space with distance function d and $x, y, z \in E$. Then

$$|d(x,y) - d(x,z)| \leq d(x,y)$$

Solution. The triangle inequality implies the following two inequalities:

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) \\ d(x,z) &\leq d(x,y) + d(y,z) \end{aligned}$$

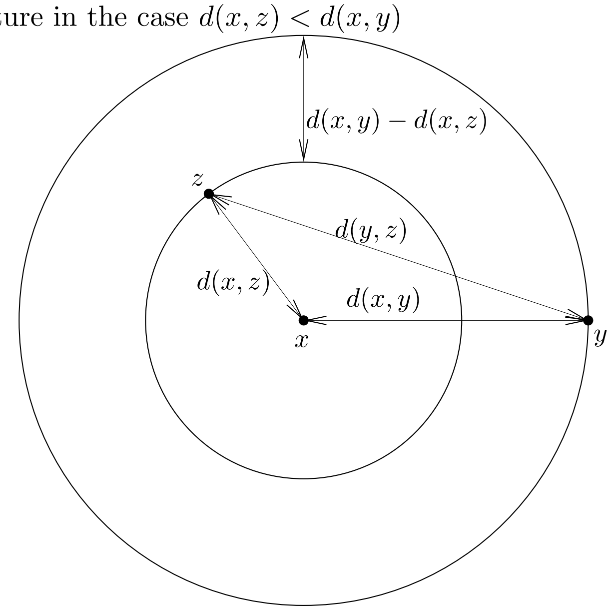
These (and the symmetry of the distance function) imply

$$\begin{aligned} d(x,y) - d(x,z) &\leq d(y,z) \\ -d(y,z) &\leq d(x,y) - d(x,z) \end{aligned}$$

which implies

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

Here is a picture in the case $d(x,z) < d(x,y)$



□

Problem 3.4. Let E be a metric space with distance function d and $x_1, x_2, \ldots, x_n$. Then

$$d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

and draw a picture in the plane showing this is reasonable.

Solution. I am assuming there was no problem doing the induction to prove the inequality. Here is the picture for $n = 6$

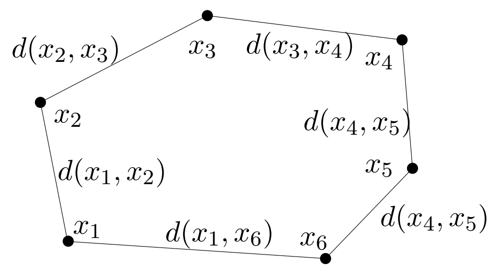


FIGURE 1. The distance of going from x_1 to x_6 is less (or equal to) going from x_1 to x_2, then to x_3, then to x_4, then to x_5 and finally to x_6. □

Problem 3.5. Show that open balls in a metric space are open set.

Solution. Let E be a metric space and $r > 0$. To show the open ball $B(a, r)$ is an open set we need to show for all $b \in B(a, r)$ there is a $r_0 > 0$ such that $B(b, r_0) \subseteq B(a, r)$. Let $r_0 = r - d(a, b)$. Then $r_0 > 0$ as $b \in B(a, r)$ and therefore $d(a, b) < r$. Let $x \in B(a, r_0)$. Then $d(x, b) < r_0$ and thus by the triangle inequality

$$d(a, x) \leq d(a, b) + d(b, x) < d(a, b) + r_0 = d(a, b) + r - d(a, b) = r,$$

and therefore $x \in B(a, r)$. This holds for all $x \in B(a, r_0)$ and so $B(b, r_0) \subseteq B(a, r)$ as required. □

Problem 3.6. Show that open intervals in $\mathbb{R}$ are open.

Solution. We have done this in class. □

Problem 3.7. Show that the compliment of a closed ball is open, and therefore closed balls are closed sets.

Solution. Let $\overline{B}(a, r)$ be a closed ball in the metric space E and let $b \in \mathcal{C}(\overline{B}(a, r))$. Then, as $b \in \mathcal{C}(\overline{B}(a, r))$ we have $d(a, b) > r$. Therefore $r_0 = d(a, b) - r$ is positive. We will show $B(b, r_0) \subseteq \mathcal{C}(\overline{B}(a, r))$. Let $x \in B(b, r_0)$. Then $d(x, b) < r_0 = d(a, b) - r$. Then by the triangle inequality

$$d(a, b) \leq d(a, x) + d(x, b) < d(a, x) + r_0 = d(a, x) + (d(a, b) - r).$$

The $d(a, b)$ terms cancel and the result can be rearranged to be

$$d(a, x) > r$$

which implies $x \in \mathcal{C}(\overline{B}(a,r))$ and as x was any element of $B(a,r_0)$ this implies $B(b,r_0) \subseteq \mathcal{C}(\overline{B}(a,r))$, showing that any element, b, of $\mathcal{C}\overline{B}(a,r)$ is the center of a ball contained in $B(a,r)$ and thus $\mathcal{C}(\overline{B}(a,r))$ is open. □

Problem 3.8. Let E be a metric space.

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Show the union $\bigcup_{i\in I} U_i$ is open.
(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Show the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Solution. Proposition 3.11 of the notes tells us that if U_1 and U_2 are open, then so is the union $U_1 \cup U_2$. Part (b) of this problem now follows by induction. (We gave a different proof in class.)

For Part (a), let $a \in \bigcup_{i\in I} U_i$, then by the definition of unions there is an $i_0 \in I$ so that $a \in U_{i_0}$. As U_{i_0} is open there is an $r_0 > 0$ such that $B(a,r_0) \subseteq U_{i_0}$. But then by basic set theory

$$B(a,r_0) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i$$

and therefore a is the center of a ball contained in $\bigcup_{i\in I} U_i$ and a was any element of $\bigcup_{i\in I} U_i$ and therefore $\bigcup_{i\in I} U_i$ is open. □

Problem 3.9. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

Solution. If $x \neq 0$, then by Archimedes' Axiom there is a natural number n so that $1/n < |x|$ and thus $x \notin U_n = (-1/n, 1/n)$. Thus $x \notin \bigcap_{n=1}^{\infty} U_n$. So $\bigcap_{n=1}^{\infty} U_n = \{0\}$. But $\{0\}$ is not open as 0 is only element of $\{0\}$ and for any $r > 0$ the ball $B(0,r) = (-r,r)$ is not contained in $\{0\}$. Therefore $\{0\}$ is not open. □

Problem 3.10. Show that in $\mathbb{R}$ that the closed intervals are closed sets.

Solution. We have done this in class. □

Problem 3.11. If E is a metric space, then every finite subset of E is closed.

Solution. Let $F = \{p_1, p_2, \ldots, p_n\}$ be a finite subset of E and $a \in \mathcal{C}F$. Then $a \neq p_j$ for $1 \leq j \leq n$ and therefore $d(a,p_j) > 0$ for each j. Therefore

$$r = \min\{d(a,p_1), d(a,p_2), \ldots, d(a,p_n)\}$$

is positive. (In fact we have shown that a finite set of real numbers always has a maximum and therefore $r = d(a,p_{j_0}) > 0$ for some $j_0 \in \{1,2,\ldots,n\}$.) But then $d(a,p_j) \geq r$ for all j and thus $p_j \notin B(a,r)$, which shows that $B(a,r) \cap F = \varnothing$ and whence $B(a,r) \subseteq \mathcal{C}(F)$. Therefore any $\mathcal{C}(F)$ point of $\mathcal{C}(F)$ is the center of a ball contained in $\mathcal{C}(F)$ and so F is open. □

Problem 3.12. Show that in the real numbers that the half open interval $[0,1)$ is neither open or closed.

Solution. We first show that $[0, 1)$ is not open. We need to find a point of $[0, 1)$ that is not the center of any ball contained in $[0, 1)$. Consider the point $0 \in [0, 1)$. For any $r > 0$ the ball $B(0, r) = (-r, r)$ will contain negative numbers, and $[0, 1)$ contains no negative numbers. Thus $[0, 1)$ is not open.

To show that $[0, 1)$ is not closed, we need to show the compliment $([0, 1)) = (-\infty, 0) \cup [1, \infty)$ is not open. Then number $1 \in \mathcal{C}([0, 1))$ and for any $r > 0$ ball $B(1, r) = (1 - r, 1 + r)$ will contain elements that are between in the interval $[0, 1)$ and therefore are not in $\mathcal{C}([0, 1))$. This thus for no $r > 0$ is $B(1, r)$ contained in $\mathcal{C}([0, 1))$ and therefore $\mathcal{C}([0, 1))$ is not open, which shows $[0, 1)$ is not closed. □

Problem 3.13. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m, n) = |m - n|$. Note that for this metric space if $m \neq n$ that $d(m, n)$ is a nonzero positive integer and thus $d(m, n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n, r)$ is the one element set $B(n, r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open.
(c) Every subset of $\mathbb{Z}$ is closed. □

Solution. (a) Let $n \in \mathbb{Z}$ and $m \in B(n, 1/2)$. Then $d(n, m) = |m - n| < 1/2$ and at $|n - m|$ is a non-negative integers the only way this is possible is if $|n - m| = 0$, that is $m = n$. Therefore in $\mathbb{Z}$ the one element set $\{n\} = B(n, 1/2)$ is an open ball and therefore and open set.

(b) Let U be any subset of $\mathbb{Z}$. Then

$$U = \bigcup_{n \in U} \{n\}$$

and each $\{n\}$ is open. Therefore U is a union of open sets and thus open.

Here is a second proof of (b). Let $n \in U$, then $B(n, 1/2) = \{n\} \subseteq U$. Therefore n is the center of a ball which is contained in U and therefore U is open.

(c) Let F be any subset of $\mathbb{Z}$. Then by Part (b) of this problem we have that its compliment in $\mathbb{Z}$, that is the set $U = \mathbb{Z} \setminus F$, is open in $\mathbb{Z}$. Therefore the compliment of F is open and thus F is closed. □

Problem 3.14. Let E be a metric space.

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Show the intersection $\bigcap_{i \in I} F_i$ is closed.
(b) Let $F_1, \ldots, F_n$ be a finite collection of closed subsets of E, show the union $F_1 \cup \cdots \cup F_n$ is closed.

Solution. This problem is why we have reviewed DeMorgan's Laws.

(a) As each F_i is closed the compliment $\mathcal{C}(F_i)$ is open. Therefore (DeMorgan) the compliment of the intersection is

$$\mathcal{C}\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} \mathcal{C}(F_i)$$

which is the union of open sets and thus open. Therefore $\bigcap_{i\in I} F_i$ is closed.

(b) This is a variant on the same theme. Each $\mathcal{C}(F_i)$ is open and therefore the compliment of the union is

$$\mathcal{C}\left(F_1 \cup \cdots \cup F_n\right) = \mathcal{C}(F_1) \cap \mathcal{C}(F_2) \cap \cdots \cap \mathcal{C}(F_n)$$

which is a finite intersection of open sets and therefore open. As the compliment of $F_1 \cup F_2 \cup \cdots \cup F_n$ is open $F_1 \cup F_2 \cup \cdots \cup F_n$ is closed. □

Problem 3.15. Let E be a metric space and $f\colon E \to \mathbb{R}$ be Lipschitz and $c \in \mathbb{R}$. Then $\{p : f(p) > c\}$ is open and $\{p : f(p) \leq c\}$ is closed.

Solution. As f is Lipschitz there is a $M > 0$ such that

$$|f(p) - f(q)| \leq Md(p, q)$$

for all $p, q \in E$. Let $g\colon E \to \mathbb{R}$ be the negative of f, that is $g(p) = -f(p)$. Then for $p, q \in E$

$$|g(p) - g(q)| = |-f(p) + f(q)| = |f(p) - f(q)| \leq Md(p, q)$$

and thus g is Lipschitz with the same constant. By the part of Proposition 3.18 that is proven in the notes (and replacing f by g) we have that $\{p : g(p) < -c\}$ is open and $\{p : g(p) \geq -c\}$ is closed. But then

$$\{p : g(p) < -c\} = \{p : -f(p) < -c\} = \{p : f(p) > c\}$$

which shows $\{p : f(p) > c\}$ is open. Also

$$\{p : g(p) \geq -c\} = \{p : -f(p) \geq -c\} = \{p : f(p) \leq c\}$$

which shows $\{p : f(p) \leq c\}$ is closed. □

Problem 3.16. Let E be a metric space and $F\colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set $\{p : f(p) = c\}$ is closed.

Solution. Just note

$$\{p : f(p) = c\} = \{p : f(p) \geq c\} \cap \{p : f(p) \leq c\}$$

and therefore $\{p : f(p) = c\}$ is the intersection of two closed sets and therefore is closed. □