Answer key to selected homework problems.

Problem 3.17 Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 1\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \geq 0, x + y \leq 1\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. □

Solution. We have seen in earlier propositions and problems that if $f \colon E \to \mathbb{R}$ is Lipschitz is then for any constant $c \in \mathbb{R}$ the sets

$$\{p : f(c) > 0\} \quad \text{and} \quad \{p : f(c) < 0\}$$

are open and the sets

$$\{p : f(c) \geq 0\} \quad \text{and} \quad \{p : f(c) \leq 0\}$$

are closed and that for any $(a, b) \in \mathbb{R}^2$ that the function

$$f(x, y) = ax + by$$

is Lipschitz (this a special case of the example at the bottom of page 51).

(a) The set $\{(x, y) \in \mathbb{R}^2 : ax + by = c\} = \{(x, y) \in \mathbb{R}^2 : ax + by \leq c\} \cap \{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$ is the intersection of two closed sets and thus closed.

(b) and (c) follow directly from the results just quoted.

(d) Write T as

$$T = \{(x, y) : x > 0\} \cap \{(x, y) : y > 0\} \cap \{(x, y) : x + y < 1\}.$$

Thus T is a finite intersection of open sets and thus open.

(e) This is the same idea S as

$$S = \{(x, y) : x \geq 0\} \cap \{(x, y) : y \geq 0\} \cap \{(x, y) : x + y \leq 1\}$$

so that S is an intersection of closed sets and thus closed. □

Problem 3.18. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Then show that $\lim_{n\to\infty} a_n = p$ also holds.

Solution. Let $\varepsilon > 0$ then there is a $N \geq 0$ such that if $n \geq N$, then $d(p_n, p) < \varepsilon$. If $n \geq N$, then also $2n \geq N$ and thus $d(a_n, p) = d(p_{2n}, p), \varepsilon$, which verifies the definition of $\lim_{n\to\infty} a_n = p$. □

Problems 3.19, 3.20, 3.21. See class notes.
Problem 3.22. Yet anther induction proof.
Problem 2.32. Let $a > 0$ and $x \geq x/4$ then

$$|\sqrt{x} - \sqrt{a}| \leq \frac{2|x-a|}{3\sqrt{a}}.$$

Solution. Start by rationalizing the numerator:

$$|\sqrt{x} - \sqrt{a}| = \left|\frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}}\right| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}}$$

But $x \geq a/4$ and therefore

$$\sqrt{x} \geq \frac{\sqrt{a}}{\sqrt{4}} = \frac{\sqrt{a}}{2}.$$

Adding $\sqrt{a}$ to this gives

$$\sqrt{x} + \sqrt{a} \geq \sqrt{a} + \frac{\sqrt{a}}{2}.$$

Taking the reciprocal:

$$\frac{1}{\sqrt{x} + \sqrt{a}} \leq \frac{1}{\sqrt{a} + \frac{\sqrt{a}}{2}}$$

Using this our equality for $|\sqrt{x} - \sqrt{a}|$ gives

$$|\sqrt{x} - \sqrt{a}| \leq \frac{|x-a|}{\sqrt{a} + \frac{\sqrt{a}}{2}} = \frac{2|x-a|}{3\sqrt{a}}.$$

□