Mathematics 554 Homework.

From *Notes on Analysis* Do Problems 4.42 3.44 3.45 3.46 3.47.
And here are some more problems.

Problem 1. Show that a bounded monotone increasing sequence of real numbers is convergent.

Problem 2. In this problem we make sense of the expression

$$\alpha = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}$$

and find its value. To start let

$$x_0 = \sqrt{2}$$

and define x_1, x_2, x_3, ... recursively by

$$\begin{aligned} x_1 &= \sqrt{2 + x_0} \\ x_2 &= \sqrt{2 + x_1} \\ x_3 &= \sqrt{2 + x_2} \\ \vdots\ & \quad \vdots \\ x_{n+1} &= \sqrt{2 + x_n} \end{aligned}$$

(a) Give the explicit formulas for x_1, x_2 and x_3. Our goal now is to show that $\lim_{n\to\infty} x_n$ exists. This will then be our definition of α.
(b) Note that $x_1 = \sqrt{2 + x_0} > \sqrt{2} = x_0$. Therefore $x_1 > x_0$. Use this as a base case of an induction to show that $\langle x_n \rangle_{n=1}^{\infty}$ is an increasing sequence. *Hint:* There are many ways to do this. One way is to use the rationalizing the numerator trick that was worked for us before. To be a little more explicit show

$$x_{n+1} - x_n = \sqrt{2 + x_n} - \sqrt{2 + x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{2 + x_n} + \sqrt{2 + x_{n-1}}}$$

and use this to do the induction step.
(c) Use induction to show $x_n < 2$ for all n.
(d) Therefore $\langle x_n \rangle_{n=1}^{\infty}$ is a bounded monotone sequence and thus is convergent. Let

$$\alpha = \lim_{n\to\infty} x_n$$

So we have made sense of what α should mean, but we still need to find its value. Justify that

$$\lim_{n\to\infty} x_{n+1} = \alpha$$

and

$$\lim_{n\to\infty} \sqrt{2 + x_n} = \alpha$$

and therefore we can take limits in

$$x_{n+1} = \sqrt{2 + x_n}$$

to get

$$\alpha = \sqrt{2 + \alpha}.$$

Solve this to find α. □

Problem 3. (Extra credit) Let $b > 0$ can your do a similar analysis to make sense of and find the value of

$$\beta = \sqrt{b + \sqrt{b + \sqrt{b + \sqrt{b + \sqrt{b + \cdots}}}}}$$

□

Problem 4. Here is an easier variant on the theme of the last couple of problems. Let $x_0 = 0$ and define a sequence by

$$x_{n+1} = \frac{2x_n}{3} + 42.$$

Show that this sequence is increasing and bounded above and find its limit. *Hint:* The increasing part should not be too hard. To get an upper bound just try some large number. If nothing else, 666 works.