Mathematics 554H Test 2 Name:______________________________

Show your work to get credit.

1. (15 points) Let E be a metric space with distance function $d(p, q)$.

(a) Let S be a subset of E. Define what it means for p to be an ***adherent point*** of S.

(b) Prove that if p is an adherent point of S that there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty} \subseteq S$ with $\lim_{n\to\infty} p_n = p$.

2. (10 points) points) Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers with $\lim_{n\to\infty} = L$. Give a ε, N proof that

$$\lim_{n\to\infty} x_n^2 = L^2.$$

3. (15 points) Define a sequence by

$$x_0 = 100$$

$$x_{n+1} = \frac{3x_n}{5} + 17 \qquad \text{for } n \geq 1.$$

(a) Show the sequence $\langle x_n \rangle_{n=0}$ is both bounded below and decreasing.

(b) Explain why $\lim_{n\to\infty} x_n$ exists.

(c) Find $\lim_{n\to\infty} x_n$.

4. (10 points points) We have shown that if $f\colon E \to \mathbb{R}$ is a Lipschitz function and $\lim_{n\to\infty} p_n = p$ in E, then $\lim_{n\to\infty} f(p_n) = f(p)$. Use this to show the set $\{p : a \leq f(x) \leq b\}$ is closed in E.

5. (10 points) Let $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ be three sequences of real numbers such that

$$x_n \leq y_n \leq z_n$$

for all n and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$$

for some L. Give a ε, N proof that

$$\lim_{n\to\infty} y_n = L.$$

6. (10 points) Let S be a subset of $\mathbb{R}^2$.

(a) Define what is means for S to be sequentially compact.

(b) Use that closed bounded subsets of $\mathbb{R}^2$ are sequentially compact to show that if A and B are closed bounded subsets of $\mathbb{R}^2$ there are points $a_0 \in A$ and $b_n \in B$ such that

$$\|a - b\| \geq \|a_0 - b_0\| \qquad \text{for all } a \in A \text{ and all } b \in B.$$