Mathematics 554H Test 3 Name:______________________________

Show your work to get credit.

1. (10 points) Define or state the following.

(a) K is a ***compact*** subset of the metric space E.

(b) K is a ***sequentially compact*** subset of the metric space E.

(c) The metric space E is connected.

(d) A condition on a subset F of the metric space E which is equivalent to F being closed. (More than one answer is possible, just chose one.)

(e) The ***Lipschitz Intermediate Value Theorem*** for a Lipschitz function $f\colon [a,b] \to \mathbb{R}$.

2. (15 points) Use the Intermediate Value Theorem (which we know is true for polynomials) to show that if $f\colon [a,b] \to [a,b]$ is a polynomial, then f has a fixed point in $[a,b]$, that is there is an $x_0 \in [a,b]$ such that $f(x_0) = x_0$. *Hint:* Consider the polynomial $g(x) = f(x) - x$. What can you say about $g(a)$ and $g(b)$?

3. (15 points) (a) Let $\mathbb{Z}$ be the integers viewed as subset of $\mathbb{R}$, that is if $m, n \in Z$ the distance between m and n is $|m - n|$. Show that the only connected subsets of $\mathbb{Z}$ are the one point sets.

(b) Let E be a connected metric space and $f\colon E \to \mathbb{Z}$ (here $\mathbb{Z}$ is the set of integers) be Lipschitz. Use the fact that Lipschitz functions map connected sets to connected to show that f is constant. *Hint:* Part (a) of this problem is relevant.

4. (10 points) Show that if K is a compact subset of the metric space E, then K is closed in E.

5. (10 points) Let $K_1 \supseteq K_2 \subseteq K_3 \supseteq K_4 \supseteq \cdots$ be a sequence of nonempty compact subsets of the metric space E. Show the intersection is nonempty. That is

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Hint: One way to do this is show that if this is false and $U_n = \mathcal{C}(K_n)$, then $\{U_n\}_{n=1}^{\infty}$ is an open cover of K_1 and show this leads to a contradiction. (You can use that compact sets are closed.)

6. (5 points) Draw a picture of two connected subset A and B of $\mathbb{R}^2$ such that $A \cap B$ is not connected (no proof required).

7. (5 Points) Which of the letters

a b c d e f c h i j k l m n o p q r s t u v w x y z

are disconnected? (Again no proof required.)

The disconnected letters are: ________________________________