Mathematics 172 Homework.

In class today we started looking at ***difference equations*** also called ***discrete dynamical systems***. Let us start with a simple example that of unconstrained population. Here is the basic model. Let N_t be the size of a population in year t. Assume that the per capita growth rate is r. That is each member of the population produces (on the average) r off spring. (Often in these cases only the females are counted as they are the ones producing offspring.) So for example if $r = 2$, then each member of the population (again on the average) produces 2 offspring each year. In biologically realistic cases usually r is not a whole number, for example the per capita of the population of the USA in 2022 was 0.38%, so that $r = .0038$.

If the per capita growth rate is r and the population in year t is N_t, then the total number of new organisms added to the population is the population size times the growth rate. That is the population the next year is

$$N_{t+1} = N_t + rN_t = (1 + r)N_t.$$

The number $1 + r$ comes up enough that it is worth giving a name. Let

$$\lambda = 1 + r.$$

This is sometime called the ***growth ratio***. Then the basic difference equation for unconstrained growth is

$$N_{t+1} = \lambda N_t \qquad \text{where} \qquad \lambda = 1 + r \tag{1}$$

and r is the per capita growth rate. When talking about unconstrained growth we assume r is a constant. For example if $r = .15$ (which corresponds to a 15% growth rate) and we have a population of size N_t in year t, then the population the next year is

$$N_{t+1} = N_t + .15N_t = (1.15)N_t.$$

Therefore if we know the population in year t that is we know N_t, then to find the population then next year we just have to multiply by 1.15.

In general if the per capita growth rate is r, then

$$N_{t+1} = N_t + rN_t = (1 + r)N_t = \lambda N_t.$$

If N_t solves (1) then knowing the population size, N_t, in year t let us compute the population size the next year, N_{t+1}, by just multiplying by the growth ratio λ. Therefore we use this starting with N_0 to get

$$\begin{aligned} N_1 &= \lambda N_0 = N_0\lambda \\ N_2 &= \lambda N_1 = \lambda N_0 \lambda = N_0\lambda^2 \\ N_3 &= \lambda N_2 = \lambda N_0 \lambda^2 = N_0\lambda^3 \\ N_4 &= \lambda N_3 = \lambda N_0 \lambda^3 = N_0\lambda^4 \\ N_5 &= \lambda N_4 = \lambda N_0 \lambda^4 = N_0\lambda^5 \\ N_6 &= \lambda N_5 = \lambda N_0 \lambda^5 = N_0\lambda^6 \\ N_7 &= \lambda N_6 = \lambda N_0 \lambda^6 = N_0\lambda^7 \end{aligned}$$

and at this point you see the pattern:

$$N_t = N_0\lambda^t. \tag{2}$$

Problem 1. A population of 20 killifish is introduced into a pond. These fish breed just one a year and only live a year. Assume that the per capita grow of the fish is .31 fish/fish.

(a) Give a formula for the number of fish after t years. *Solution:* If N_t is the population size, then the growth ratio is $\lambda = 1 + .31 = 1.31$ and $N_0 = 20$. Therefore $N_t = N_0\lambda^t = 20(1.31)^t$.

(b) How long until there are 2,000 fish? *Solution:* We need to solve $N_t = 20(1.31)^t = 2{,}000$. The solution is

$$t = \frac{\ln(2{,}000/20)}{\ln(1.31)} = 17.05\text{years}.$$

So to guarantee 2,000 should wait to year 18. □

Proposition 1. *In a different pond* 50 *of the killifish are released. The population is counted five years later and there are* 213 *of the fish. What is the per capita growth rate of the population?* Solution: *If* N_t *is the population size and* λ *is the growth ratio, then* $N_t = 50\lambda^t$. *To find* λ *set*

$$N_5 = 50\lambda^5 = 231.$$

This gives

$$\lambda = \left(\frac{231}{50}\right)^{1/5} = 1.3581$$

and therefore the per capita is $r = \lambda - 1 = .3581$. □

We are also going to want to look at more general equations of the form

$$N_{t+1} = f(N_t).$$

That is where if we know the population, N_t, in year t, then there is a function, f, so that we can find the population size in the next year by applying f to N_t.

Here is an example. Assume that

$$N_{t+1} = N_t + .5N_t\left(1 - \frac{N_t}{100}\right) \qquad N_0 = 50.$$

Then

$$N_1 = N_0 + .5N_0\left(1 - \frac{N_0}{100}\right) = 50 + .5(50)\left(1 - \frac{50}{100}\right) = 62.5$$
$$N_2 = N_1 + .5N_1\left(1 - \frac{N_1}{100}\right) = 62.5 + .5(62.5)\left(1 - \frac{62.5}{100}\right) = 74.219$$
$$N_3 = N_2 + .5N_2\left(1 - \frac{N_2}{100}\right) = 74.219 + .5(74.219)\left(1 - \frac{74.219}{100}\right) = 83.78$$
$$N_4 = N_3 + .5N_3\left(1 - \frac{N_3}{100}\right) = 83.7860 + .5(83.7860)\left(1 - \frac{83.7860}{100}\right) = 90.57854$$

and we can go on indefinitely.

Problem 2. If

$$P_{t+1} = \frac{4P_t}{1 + .1(P_t)^2} \qquad P_0 = 3$$

find P_1, P_2 and P_3. *Solution:* $P_1 = 6.3158$, $P_2 = 5.0639$, and $P_3 = .6829$. □