Mathematics 172 Homework.

These notes/homework give some more practice on cobwebbing and also using your calculator to find equilibrium points and determine if they are stable. If you are not sure how this works let me again recommend the video:
https://www.youtube.com/watch?v=nxcKh36rep0.

Read over the last homework and make sure you understand about equilibrium points. One of the things we mentioned in class is that if N_* is an equilibrium point of

$$N_{t+1} = f(N_t)$$

then it is stable if $|f'(N_*)| < 1$ and unstable if $|f'(N_*)| > 1$.

Problem 1. Let $r, K > 0$. Then the discrete logistic with pre capita growth rate of r and carrying capacity K is

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) = f(N)$$

where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right)$$

We wish to find the equilibrium points and see if they are stable.

(a) We first look at the special case where $r = .2$ and $K = 100$. Find the equilibrium points and determine if they are stable. *Solution:* In this case we wish to solve

$$f(N) = N + .2N\left(1 - \frac{N}{100}\right) = N.$$

This reduces to

$$.2N\left(1 - \frac{N}{100}\right) = 0$$

and we see the equilibrium are

$$N_* = 0, 100.$$

Now compute the derivative of f. To start it is a bit easier if we first rewrite f a bit.

$$f(N) = N + .2N - \frac{.2N^2}{100}.$$

This

$$f'(N) = 1 + .2 - \frac{.4N}{100}.$$

At $N_* = 0$ we have

$$f'(0) = 1 + .2 - \frac{.4(0)}{100} = 1.2 > 1$$

and therefore $N_* = 0$ is unstable. At $N_* = 100$ we have

$$f'(100) = 100 + .2 - \frac{.4(100)}{100} = .8$$

which shows that this point is also stable.

(b) Now do the general case where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right) f(N)$$

Solution: The equilibrium points are $N_* = 0$ and $N_* = K$ A calculation like the ones done above yield that

$$f'(0) = 1 + r > 1$$

and so for the logistic equation $N_* = 0$ is always unstable. We also have

$$f'(K) = 1 - 2r$$

This in this case $N_* = K$ is stable when $0 < r < 2$ (which implies $|1-2r| < 1$ and it is unstable when $2 < r$ (which implies $|1-2r| > 1$). □

Putting this all together gives

Theorem 1. *For the discrete logistic equation with per capita growth rate r and carrying capacity K,*

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right)$$

there are two equilibrium points $N_* = 0$ and $N_* = K$.

- *$N_* = 0$ is always unstable for the discrete logistic.*
- *$N_* = K$ is stable for $0 < r < 2$ and unstable for $r > 2$.*

Here are some problem for finding equilibrium points in discrete dynamical and determining if they are stable using your calculator.

Problem 2. For the dynamical system

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t^{1.5}}{100}\right)$$

Find the equilibrium points and determine if they are stable.

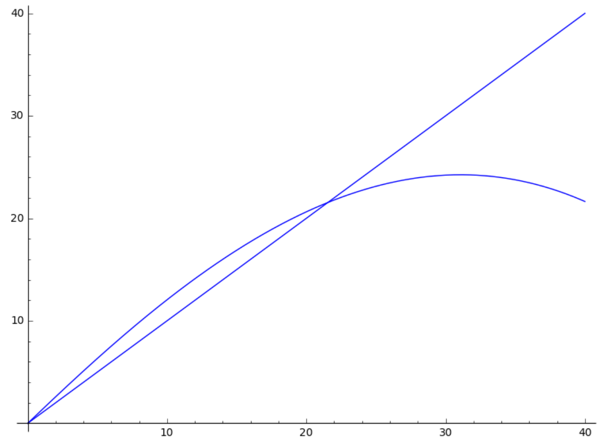
Solution: First enter

```
\Y1=X+.3X(1-X^(1.5)/100)
\Y2=X
```

And by some trial and error I found that good size for the window is

```
Xmin=0
Xmax=40
```

Now plot by using `ZOOM` and then `0:ZoomFit` The result should look something like:



From the picture we see that there are two equilibrium points.

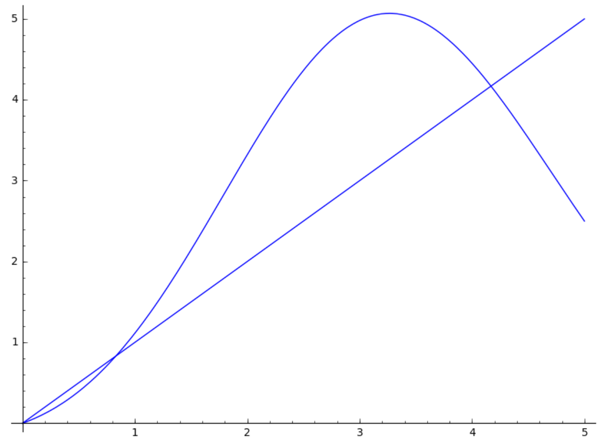
To find them use `2nd CALC 5:intesect` It will now ask your `First curve`. Just hit `ENTER`. It will now ask `Second curve`. Again just hit `ENTER`. The next (and last) question is `Guess?`. This time move the cursor to be as close as possible to the intersection point you want to find and hit `ENTER`. It you moved the cursor to 0 (which is clearly and equilibrium point from the picture) we get that `x=0` and `Y=0` is an intersection point. To find if this is stable we hit `2nd CALC 6:dy/dx` which will give us the value of the derivative of the `\Y1` curve at the current x value, which is $x = 0$. This will tell use that $f'(0) = 1.3$. As $|f'(0)| = 1.3 > 1$ the point $P_* = 0$ is unstable.

To find the second equilibrium point go through the steps above with the difference that when you are ask `Guess?`, you move the cursor to get as close as you can to the other point of intersection. This time you will get that `X=21.544347` and that `Y` has the same value. Therefore $P_* = 21.544347$ is the second equilibrium point. Now use the calculator to find the value of $f'(21.544347) = .55$ so this point is stable.

Problem 3. For the dynamical system

$$N_{t+1} = .5N_t e^{N_t - .2N_t^2}$$

(a) Plot $y = f(x) = .5xe^{x-.2x^2}$ and $y = x$ on your calculator for $0 \le x \le 5$.
Solution: Your picture should look like:



(b) So we see there are three equilibrium points. Now find them and determine if they are stable or unstable.

Solution: The first is $P_* = 0$. As $f'(0) = .500$ this one is stable.

The second is $P_* = .83138857$ and at this point $f'(.83138857) = 1.5549057$ so this one is unstable.

The third is $P_* = 4.1686114$ and at this point $f'(4.1686114) = -1.759351$ so this one is unstable.

Problem 4. Let

$$f(P) = \frac{5 + 20P}{1 + P^2}$$

and consider the discrete dynamical

$$P_{t+1} = f(P_t).$$

(a) Graph $y = f(x)$ and $y = x$ for with $0 \le x \le 10$ and use the calculator to find the where these graphs intersect. *Solution:* There is only one point of intersection and it is $P_* = 4.48495684796404$.

(b) Use the calculator to find $f'(P_*)$. *Solution:* $f'(P_*) = -0.958078670794696$.

(c) Is P_* stable or unstable? *Solution:* since $|f'(P_*)| = 0.958078670794696 < 1$ the point is stable.

Problem 5. One of my summer jobs was working in a pet shop where there was a bit of a bug problem. Assume that the population of roaches in the shop grows with a discrete logistic rule with a per capita growth rate of $r = .8$ (roaches/week)/roach and a carrying capacity of $K = 100$ roaches. Let N_t be the number of roaches in week t.

(a) What is the equation relating N_t and N_{t+1}?

(b) To control the bugs the owner gave a pair of Tokay geckos the run of the shop. Together the lizards ate the roaches at the rate of 15 bugs/week. What is the new equation satisfies by N_t and N_{t+1}?

(c) What happens to the roach population after the geckos are released?

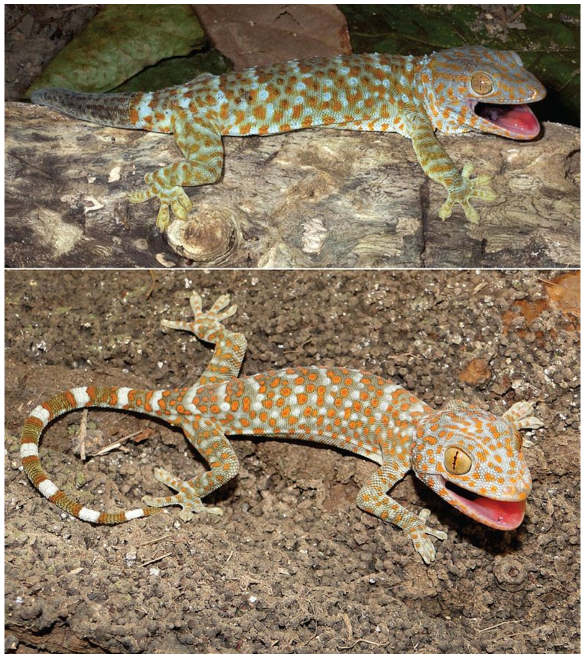


FIGURE 1. Adult and juvenile Tokay geckos.

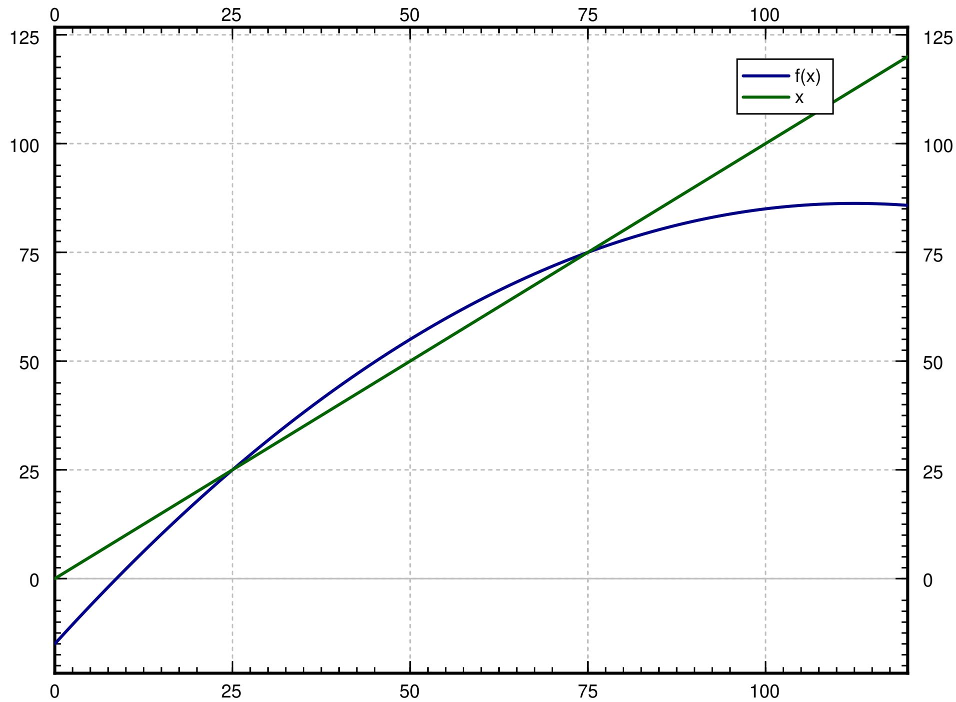
Solution. (a) This is just the discrete logistic equation:

$$N_{t+1} = N_t + .8N_t\left(1 = \frac{N_t}{100}\right).$$

(b) The new equation is

$$N_{t+1} = N_t + .8N_t\left(1 = \frac{N_t}{100}\right) - 15.$$

(c) Plot \Y1= X +.8X(1-X/100)-15 and \Y2=X to get a graph that looks like the one below



The points of intersection are $N_* = 25$ and $N_* = 75$. From the graphs we see (do cobwebbing) that 25 is unstable and 75 is stable. Therefore the roach population stabilizes at 75. □