Mathematics 172 Homework, October 20, 2023.

When talking about rate equations there is important topic we skip, which is Euler's method. To start recall that if $f(x)$ is a function, then the official definition of the derivative of f at $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Put less formally this means that if h is very close to 0 then

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

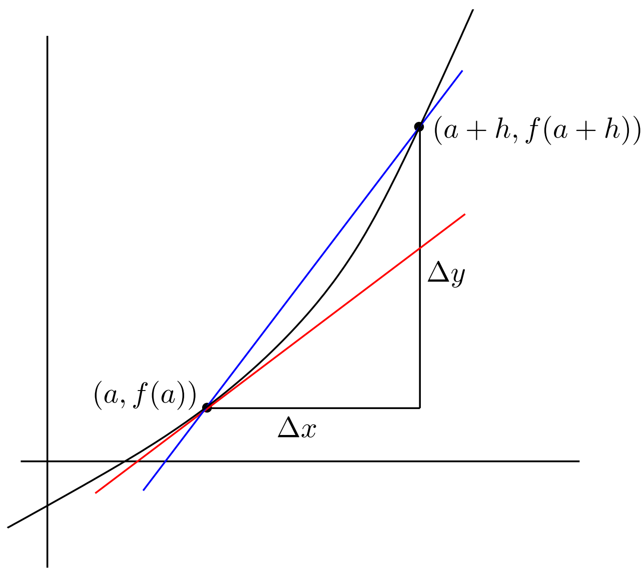

FIGURE 1. The slope of the line (shown in blue) though $(a, f(a))$ and $(a+h, f(a+h))$ the rise over run. In this case the rise is $\Delta y = f(a+h) - f(a)$ (the difference in the y values) and the run is $\Delta x = (a+h) - a = h$ (the difference in the x values.) So the slope is

$$\text{slope of blue line } = \frac{f(a+h) - f(a)}{h}.$$

When h is very small this line will be very close to the line tangent to the graph at $(a, f(a))$ (tangent line shown in red).

1. As an example of this let see what happens for the function $f(x) = x^2 + x$ at the point $a = 1$. The derivative is

$$f'(x) = 2x + 1$$

and therefore

$$f'(1) = 2(1) + 1 = 3.$$

(a) For this function compute $\dfrac{f(a+h)-f(a)}{h}$ for $a = 1$ and the three values $h = .1$, $h = .01$, and $h = .001$. *Solution:* For $h = .1$ it is 3.1, for $h = .01$ it is 3.01 and for $h = .001$ is it 3.001. So as h is getting smaller the value $\dfrac{f(a+h)-f(a)}{h}$ is getting closer to $f'(1) = 3$. □

If we take the basic formula

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

and multiply by h

$$f'(a)h \approx f(a+h) - f(a)$$

and then solve for $f(a+h)$ we get

$$\boxed{f(a+h) \approx f(a) + f'(a)h.}.$$

This lets us approximate $f(a+h)$.

Example 1. If $f(3) = 17$ and $f'(3) = 2$ estimate $f(3.05)$. In this case we have $a = 3$ and $h = .05$. So

$$f(3.05) \approx f(3) + f'(3)(.05) = 17 + 2(.05) = 17.1,$$

□

2. If $P(2) = 5$ and $P'(2) = -.21$ estimate $P(2.1)$.

Solution: This time $a = 2$, and $h = .1$ and we have

$$P(2.1) \approx P(2) + P'(2)(.1) = 5 + (-.21)(.1) = 4.979$$

Now on to rate equations. For the equation logistic equation

$$P'(t) = .15P(t)\left(1 - \frac{P(t)}{100}\right)$$

Suppose we know

$$P(5) = 96$$

and we would like to estimate $P(5.1)$. From our basic approximation formula we know

$$P(5.1) \approx P(5) + P'(5)(.1).$$

We are given that $P(5) = 96$. To compute $P'(5)$ use the rate equation

$$P'(5) = .15(96)\left(1 - \frac{96}{100}\right) = 0.576$$

We can now estimate $P(5.5)$

$$P(5.5) \approx P(5) + P'(5)(.5) \approx 96 + (.576) * (.5) = 96.288$$

Here are some practice problems.

3. If N satisfies the rate equation

$$\frac{dN}{dt} = .05N\left(1 - \frac{N}{2{,}000}\right)$$

and $N(50) = 2{,}010$ estimate $N(50.2)$ and $N(152)$ *Solution:* $N(50.2) \approx 2009.8995$ and $N(152) \approx 2{,}000$. (The reason $N(152) \approx 2{,}000$ is that by the time $t = 152$ the population size will have stabilized at the carrying capacity of $K = 2{,}000$.)

4. If P satisfies

$$\frac{dP}{dt} = .7P\left(1 - \frac{P}{77}\right) \qquad P(10) = 75$$

and estimate $P(10.5)$ and $P(105)$. *Solution:* $P(10.5) \approx 75.6818181818$ and $P(102) \approx 77$.