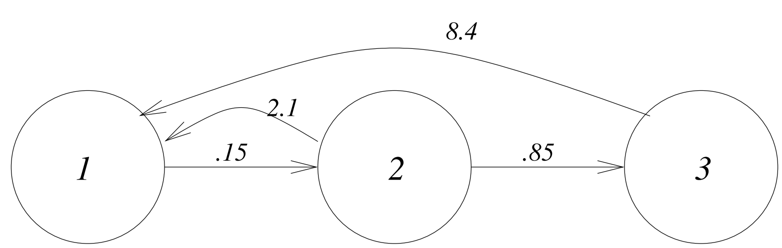
Mathematics 172 Homework, November 16, 2023.

Several types of plant are biennial. That is they live for two years and produce seeds in the second year. Examples are onion, cabbage, parsley, silverbeet, Black-eyed Susan, and carrot. For some of this plant breeders and produced varieties that will produce some small number of seeds in their first year. In the following diagram we have three stages for a type of onion. The first stage is seedlings. The second is juvenile, that is plants that are one year old, and the third stage is adults, plants that are two years old. The plant do not live to a third year.

In this loop diagram the .15 is the proportion of seedlings that survive to be juveniles, the .85 is the proportion of juveniles that survive to be adults, the 2.1 is the average number of seedlings produced by a juvenile and the 8.4 is the average number of seedlings produced by an adult.



If in some year which call year 0, we find there are 98 seedlings, 15 juveniles, and 12 adults. Then next year is year 1, the year after than year 2, etc. Compute the following:

Number of seedlings in year 1: ________

Number of juveniles in year 1: ________

Number of adults in year 1: ________

Number of seedlings in year 2 ________

Number of juveniles in year 2 ________

Number of adults in year 2 ________

Number of seedlings in year 3 ________

Number of juveniles in year 3 ________

Number of adults in year 3 ________

Solution:

In the second year

$$\text{Number of seedlings} = 15(2.1) + 12(8.4) = 132.3$$
$$\text{Number of juveniles} = 98(.15) = 14.7$$
$$\text{Number of adults} = 15(.15) = 2.25$$

In the third year

$$\text{Number of seedlings} = 137.97$$
$$\text{Number of juveniles} = 19.845$$
$$\text{Number of adults} = 12.495$$

In the fourth year

$$\text{Number of seedlings} = 146.6325$$
$$\text{Number of juveniles} = 20.6955$$
$$\text{Number of adults} = 16.86825$$

We now give the general set up for age structured population growth. To keep the notation within bounds we will first discuss a population with four stages. An example might be a frog that lives for four years. Then stage 1 would be tadpoles, stage two juveniles, stage 3 adults, and stage 4 elderly adults. We will assume that only the stage 3 and stage 4 individuals reproduce. Often in these models only the females are counted. Let

$$n_j(t) = \text{number of individuals in stage } j \text{ in year } t$$

We will put all the population sizes in one package like this:

$$\vec{N}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \end{bmatrix}.$$

This package is called a ***vector***.

We wish to understand how the numbers in each stage in year t, determine the numbers in the next year, that is year $t+1$.

For stages 2, 3, and 4 this is easy to understand. The only way that an individual is in stage 2 in year $t+1$ is if it was in stage 1 in year t and survived to the next year. So if p_1 is the proportion of stage 1 individuals that survive to stage 2, then we have $n_2(t+1) = p_1 n_1(t)$. In general if we let

$$p_j = \text{proportion of stage j individuals that survive to the next year}$$

then we have

$$\begin{aligned} n_2(t+1) &= p_1 n_1(t) \\ n_3(t+1) &= p_2 n_2(t) \\ n_4(t+1) &= p_3 n_3(t) \end{aligned}$$

(Some people use s_j rather than p_j.)

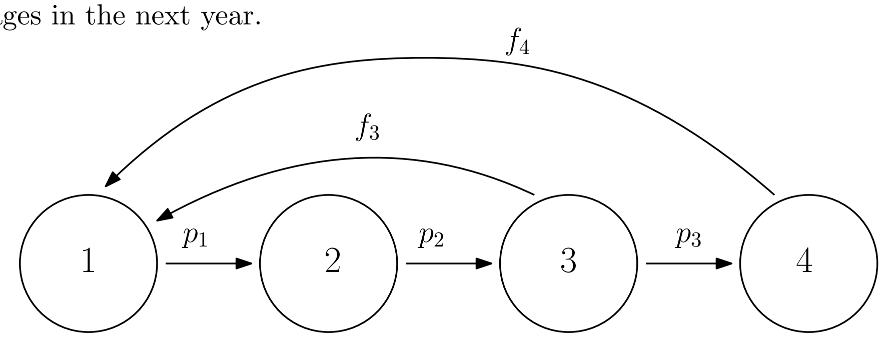
To become a stage 1 individual in year $t+1$ an individual most be the offspring of an individual who was in stage 3 or stage 4 in year t. Let

$$f_j = \text{average number of offspring of a stage j individual that survive to stage 1.}$$

This is the ***fecundity*** of the stage j individuals. (Some people use b_j where b_j can be thought of as a birth rate.) In our set up with just 4 stages the only stages which contribute to the next years stage 1 are stages 3 and 4 and we have

$$n_1(t+1) = f_3 n_3(t) + f_4 n_4(t).$$

The following loop diagram summarizes which stages contribute to the which stages in the next year.



We and write these equation for the $n_j(t)$ as one vector equation

$$\begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+3) \\ n_4(t+1) \end{bmatrix} = \begin{bmatrix} f_3 n_3(t) + f_4 n_4(t) \\ p_1 n_1(t) \\ p_2 n_2(t) \\ p_3 n_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & f_3 & f_4 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \end{bmatrix} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \end{bmatrix}.$$

where the square array

$$L = \begin{bmatrix} 0 & 0 & f_3 & f_4 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \end{bmatrix}$$

is a ***matrix***. With this notation this reduces to the one matrix equation.

$$\vec{N}(t+1) = L\vec{N}(t).$$

If you know about matrix multiplication (and do not worry if you do not) then we can solve this in pretty much the same way that we did for unconstrained population. That is given $\vec{N}(0)$ we can compute

$$\begin{aligned}
\vec{N}(1) &= L\vec{N}(0) \\
\vec{N}(2) &= L\vec{N}(1) = LL\vec{N}(0) = L^2\vec{L}N(0) \\
\vec{N}(3) &= L\vec{N}(2) = LL^2\vec{N}(0) = L^3\vec{L}N(0) \\
\vec{N}(4) &= L\vec{N}(3) = LL^3\vec{N}(0) = L^4\vec{L}N(0) \\
\vec{N}(5) &= L\vec{N}(4) = LL^4\vec{N}(0) = L^5\vec{L}N(0) \\
\vec{N}(6) &= L\vec{N}(5) = LL^5\vec{N}(0) = L^6\vec{L}N(0) \\
\vec{N}(7) &= L\vec{N}(6) = LL^6\vec{N}(0) = L^7\vec{L}N(0) \\
\vec{N}(8) &= L\vec{N}(7) = LL^7\vec{N}(0) = L^8\vec{L}N(0) \\
\vec{N}(9) &= L\vec{N}(8) = LL^8\vec{N}(0) = L^9\vec{L}N(0).
\end{aligned}$$

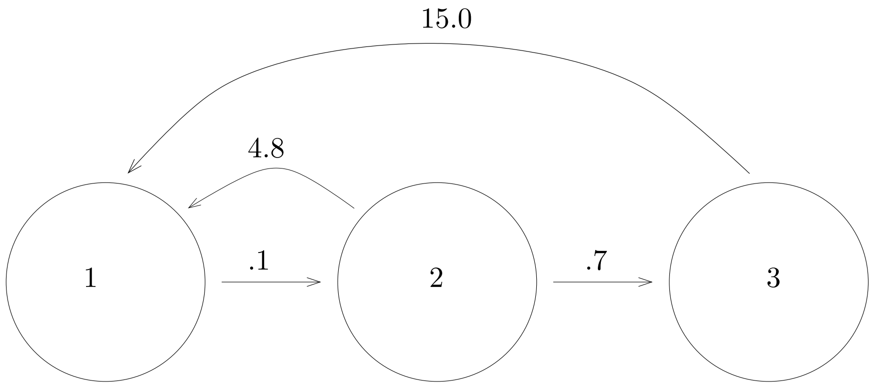
At this point you see the pattern that

$$\vec{N}(t) = L^t\vec{N}(0).$$

Fortunately we do not have to multiply the matrices by hand, or even know exactly what matrix multiplication is, because the calculator can do this for us. The matrix, L, above is the ***Leslie matrix*** of the model, named after Patrick Holt Leslie who seems to have been the first to use this model for aged structured population growth.

Before going on to how to use the calculator here are some problems about going back and fourth between loop diagrams and the Leslie matrix.

1. A population of weeds in a back yard has three stages. The first is seedling, the second is juvenile, and the third is adult. The life history of this population is summarized by the loop diagram.



(a) What does the number 4.8 mean?
(b) What does the number .7 mean?
(c) What is the Leslie matrix for this diagram?
(d) If we start with 15 seedlings, 3 juveniles, and 2 adults then how many are in each stage the next year? The second year?

Solution: (a) That on the average each individual juvenile produces 4.8 offspring that live a year to be seedlings.
(b) That the proportion of juveniles that live to be adults is .7.
(c) The Leslie matrix is

$$L = \begin{bmatrix} 0 & 4.8 & 15.0 \\ .1 & 0 & 0 \\ 0 & .7 & 0 \end{bmatrix}$$

(d) Let $\vec{N}(0)$ be the matrix

$$\vec{N}(0) = \begin{bmatrix} 15 \\ 3 \\ 2 \end{bmatrix}$$

In your calculator enter $[A] = L$ (that is enter the numbers for L into the matrix $[A]$ of your calculator. Enter $[B] = \vec{N}(0)$ with $\vec{N}(0)$ as above. Then your calculator should give

$$L\vec{N}(0) = [A][B] = \begin{bmatrix} 44.4 \\ 1.5 \\ 2.1 \end{bmatrix}$$

Thus next next year there will be 44.4 in stage 1, 1.5 in stage 2, and 2.1 in stage 3.

To find the number in each stage for the second year compute

$$\vec{N}(2) = L^2\vec{N}(0) = [A]^2[B] = \begin{bmatrix} 38.7 \\ 4.44 \\ 1.05 \end{bmatrix}$$

For the tenth year compute

$$\vec{N}(10) = L^{10}\vec{N}(0) = [A]^{10}[B] = \begin{bmatrix} 146.95 \\ 12.00 \\ 6.6 \end{bmatrix}$$

(e) Doing calculations just as in part (d) we find:

$$\vec{N}(20) = \begin{bmatrix} 689.85 \\ 58.81 \\ 35.58 \end{bmatrix} \qquad \text{and} \qquad \vec{N}(21) = \begin{bmatrix} 815.95 \\ 68.98 \\ 41.17 \end{bmatrix}$$

For year $t = 20$ the total number is $N(t) = 689.85 + 58.81 + 35.58 = 784.34$. Therefore the proportion in each class is

$$\text{Proportion in stage 1} = \frac{689.85}{784.34} = 0.879529$$
$$\text{Proportion in stage 2} = \frac{58.81}{784.34} = 0.074991$$
$$\text{Proportion in stage 3} = \frac{35.58}{784.34} = 0.045365$$

For the year $t = 21$ the total number is $N(21) = 815.95 + 68.98 + 41.17 = 926.10$ and this time the proportion in each class is

$$\text{Proportion in stage 1} = \frac{815.95}{926.10} = 0.881057$$
$$\text{Proportion in stage 2} = \frac{68.98}{926.10} = 0.074489$$
$$\text{Proportion in stage 3} = \frac{41.17}{926.10} = 0.0444530$$