Mathematics 172 Homework, September 5, 2023.

We are now going to review some calculus. Recall that if $y = f(x)$ is a function then the derivative $f'(a)$ is the slope of the tangent line to the graph at the point $(a, f(a))$.

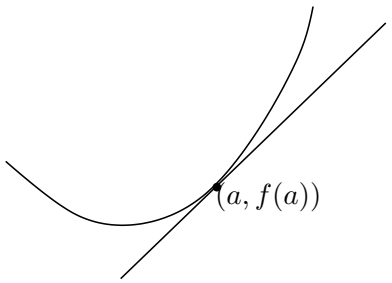

FIGURE 1. The derivative is the slope of the tangent line.

Here are some basic facts we will be using.

Proposition 1. *If the derivative of f is positive on an interval $a \leq x \leq b$, then f is increasing on (a, b).*

Likewise

Proposition 2. *If $f'(x) < 0$ on (a, b), then f is decreasing on $a \leq x \leq b$.*

Proposition 3. *If the derivative $f'(x) = 0$ on all of (a, b), then f is a constant on $a \leq x \leq b$.*

Problem 1. Show that if $f'(x) = 2$ for all x, then $f(x) = -2x + C$ for some constant C.

Solution. Let $y = f(x) - 2x$. Then the derivative of y is

$$y' = f'(x) - (2x)' = 2 - 2 = 0$$

Therefore y is constant, say $y = C$. Then $y = f(x) - 2x = C$. Solve this for $f(x)$ to get $f(x) = 2x + C$. □

We recall some derivative formulas (a, n, r are constants).

$y = ax^n$	$y' = nax^{n-1}$	
$y = ae^x$	$y' = ae^x$	
$y = ae^{rx}$	$y' = are^{rx}$	
$y = ab^x$	$y' = a\ln(b)b^x$	
$y = uv$	$y' = u'v + uv'$	(product rule)
$y = f(g(x))$	$y' = f'(g(x))g'(x)$	(chain rule)

Problem 2. Let $f(x)$ be a function that satisfies $f'(x) = f(x)$.

(a) Let $y = e^{-x}f(x)$ and use the product rule to show

$$y' = -e^{-x}f(x) + e^{x}f'(x).$$

(b) Use $f'(x) = f(x)$ to show $y' = 0$ for all x.
(c) Because y' is zero, we have that $y = C$ for some constant C.
(d) Finally conclude that $f(x) = Ce^{x}$.

Problem 3. Use a variant of the argument in the previous problem to show that if $f'(x) = 2f(x)$, then for some constant, C, $f(x) = Ce^{2x}$. *Hint:* Let $y = e^{-2x}f(x)$ and show $y' = 0$ and therefore that y is constant.

Problem 4. Use the same ideas to show that if $f'(x) = -3f(x)$ then $f(x) = Ce^{-3x}$ for some constant C.

Problem 5. Let y satisfy the differential equation $y' = \dfrac{1+x^2}{1+y^2}$ and assume $y(0) = 3$. Explain why $y(17) > 3$. *Hint:* Is the derivative of y positive or negative? If it is positive what does this say about y?

Problem 6. Find the following derivatives.

$$P = 100e^{.15t} \qquad \frac{dP}{dt} =$$

$$W = .45r^3 \qquad \frac{dW}{dr} =$$

$$y = 3x^3e^{2x} \qquad \frac{dy}{dx} =$$

$$f(t) = 4(1+t^3)^4 \qquad f'(t) =$$