Mathematics 172 Homework, September 18, 2023.

What we have just shown is that if $P = P(t)$ is the size of a population and the intrinsic growth rate

$$r = \frac{1}{P}\frac{dP}{dt}$$

is constant, then population growth is exponential:

$$P(t) = P_0 e^{rt}.$$

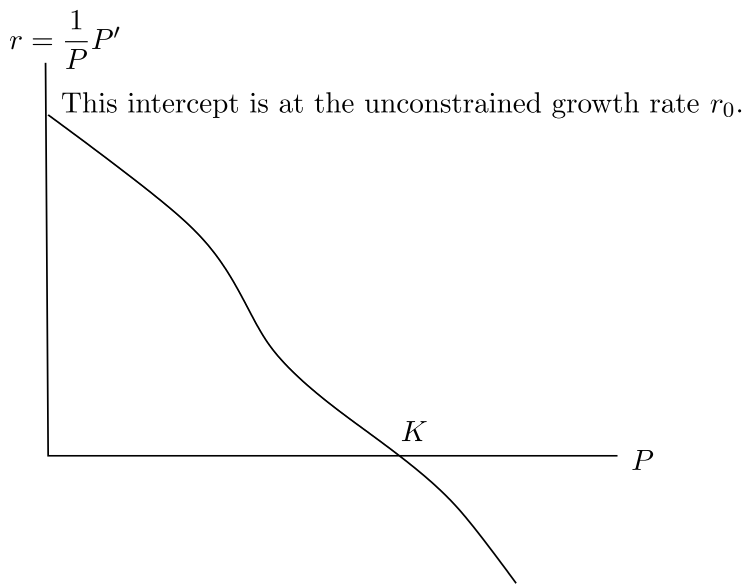
While in the short term this is fine, in the long run it either predicts that the population becomes infinite (when $r > 0$) of dies off (when $r < 0$). This is not what we see in nature.

An idea that has lead to reasonable models is to let $r = r(P)$ depend on the population size. The definition of r is the total rate change, dP/dt, divided by the population size, P. So it is in some sense the rate of change per unit of population size.

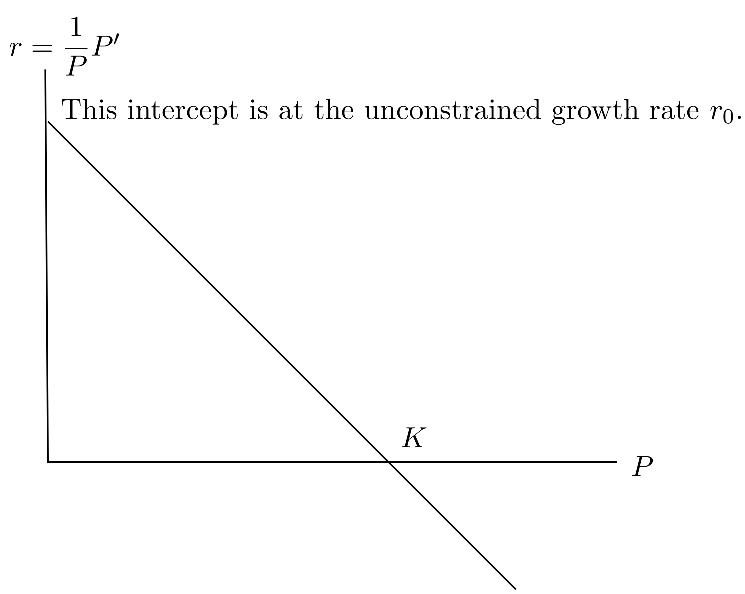
Logistic Growth. The first variant will consider is that there is a population size called the ***carrying capacity*** and denoted by K. This is to satisfy

- If $P > K$ the population is over crowded and so the intrinsic growth rate $r = r(P)$ becomes negative. That is the population size decreases.
- If $0 < P < K$ the intrinsic growth rate $r = r(P)$ is positive and so the population size is increasing.
- When P is close to zero, the population grows as if unconstrained.

That is as a function of P we want a function that looks like



The easiest choice of such a function is a straight line:



The equation of this line is

$$r(P) = r_0\left(1 - \frac{P}{K}\right).$$

Using the definition of r we then have for our rate equation for logistic growth:

$$\frac{1}{P}\frac{dP}{dt} = r_0\left(1 - \frac{P}{K}\right)$$

which can be rewritten is

$$\frac{dP}{dt} = r_0 P\left(1 - \frac{P}{K}\right).$$

From now on we will drop the subscript on r_0 and just write

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

This is the ***logistic equation*** and you should have in memorized for the rest of the term.

1. If $r = .15$ and $K = 100$, the equation is

$$\frac{dP}{dt} = .15P\left(1 - \frac{P}{K}\right)$$

(a) Find the stationary solution(s).

(b) Make a graph showing the stationary solution(s) along with the solutions with $P(0) = 75$ and $P(0) = 125$.

(c) If $P(0) = 75$ estimate $P(123)$. If $P(0) = 125$ estimate $P(231)$.

2. Now do the general case: For

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

(a) What are the stationary solution(s)

(b) If $P(0) = 17$ estimate $P(1{,}000)$.

(c) More generally if $P(t)$ is any solution to the logistic equation, with $P(0) > 0$ what can you say about $P(t)$ when t is a very large number?