Mathematics 554 Homework.

Let E be a set and $\mathcal{U}$ a collection of subsets of E. This if $U \in \mathcal{U}$, then $U \subseteq E$. Then the union of $\mathcal{U}$ is

$$\bigcup \mathcal{U} = \{x \in E : x \in U \text{ for at least one } U \in \mathcal{U}\}.$$

The intersection is

$$\bigcap \mathcal{U} = \{x \in E : x \in \text{ for all } U \in \mathcal{U}\}.$$

This is just notation for ideas you already know. For example

$$\bigcup \{U_1, U_2, U_3\} = U_1 \cup U_2 \cup U_3$$

and

$$\bigcap \{U_1, U_2, U_3\} = U_1 \cap U_2 \cap U_3$$

Problem 1. Let E be a metric space, $p \in E$ and $r_1, r_2, \ldots, r_n$ positive real numbers. Let $\mathcal{U}$ be the collection of open balls

$$\mathcal{U} = \{B(p, r_1), B(p, r_2), \ldots, B(p, r_n)\}.$$

Let $r_{\max} = \max(r_1, r_2, \ldots, r_n)$ and $r_{\min} = \min(r_1, r_2, \ldots, r_n)$. Show

$$\bigcup \mathcal{U} = B(p, r_{\max}), \qquad \bigcap \mathcal{U} = B(p, r_{\min})$$

□

Problem 2. Let E be a metric space and $p \in E$. Let R be a set of positive real numbers which is bounded above and set

$$\mathcal{U} = \{B(p, r) : r \in R\}.$$

Let

$$r_0 = \inf R, \qquad r_1 = \sup R.$$

(a) Show

$$\bigcup \mathcal{U} = B(p, r_1).$$

(b) Given an example where

$$\bigcap \mathcal{U} \neq B(p, r_0).$$

Hint: With $E = \mathbb{R}$, $p = 0$, and $R = (1, 2)$ then

$$\bigcap \mathcal{U} = \bigcap_{1<r<2} (-r, r).$$

□

Definition 1. Let E be a metric space and $S \subseteq E$. Then $\mathcal{U}$ an ***open cover*** of S if and only if

(a) Each $U \in \mathcal{U}$ is an open subset of E.
(b) $S \subseteq \bigcup \mathcal{U}$.

Here is an equivalent definition that some people find easier to work with.

Definition 2. Let E be a metric space and $S \subseteq E$.

(a) Each $U \in \mathcal{U}$ is an open subset of E.
(b) For each $s \in S$, there is $U \in \mathcal{U}$ with $s \in U$.

The following problems are just definition chases, but involve ideas that will be used later.

Problem 3. Let $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$. Show the following are equivalent
(a) $\mathcal{U}$ is an open cover of $\mathbb{R}$.
(b) Archimedes' Axiom. $\square$

Problem 4. Variants of this will come up often. Let S be a subset of the metric space E. For each $p \in S$ let $r_p > 0$ be a positive number. Show $\mathcal{U} = \{B(p, r_p) : p \in S\}$ is an open cover of S. $\square$

Problem 5. Let E be a metric space and $S \subseteq E$. Let $p \in E$. Show $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S. $\square$

Problem 6. Let S be a set that has a finite open cover $\mathcal{U}$. (That is $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ is finite.) Assume that for each $U \in \mathcal{U}$ that $U \cap S$ is finite. Then S is finite. (Do not be tempted to make this hard, it should just be a few sentences.) $\square$

Problem 7. Let S be a subset of the metric space E that has has the property that if $\mathcal{U}$ is an open cover of S, then $\mathcal{U}$ has a finite subset $\mathcal{U}_0$ which is also a cover of S. Show that S is bounded. *Hint:* Pick any $p \in E$ then we have see, Problem 5, that $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S and after taking the finite subset of $\mathcal{U}$ Problem 1 may be useful.

Let me complete the proof of the theorem by Cantor we discussed on Wednesday. Recall that the ***power set***, $\mathcal{P}(S)$, of a set S is the set of all its subsets. That is

$$\mathcal{P}(S) = \{A : A \subseteq S\}.$$

Theorem 3 (Cantor)**.** *Let S be a set, then there is no surjective function $f\colon S \to \mathcal{P}(S)$.*

Proof. Given $f\colon S \to \mathcal{P}(S)$ we want to show f is not onto. We can even explicitly define an element of $\mathcal{P}(S)$ that is not in the image of f. Let

$$B = \{s \in S : s \notin f(s)\}.$$

(As a check that this makes sense as $f\colon S \to \mathcal{P}(S)$ we have that $f(s)$ is a subset of S. So asking if $s \in f(s)$ makes sense.) **Claim:** B is not in the image of f. For if B is in the image of f, then $B = f(s)$ for some $s \in S$. But then, using $B = f(s)$ and the definition of B,

$$s \in B \quad \Longleftrightarrow \quad s \in f(s) \quad \Longleftrightarrow \quad s \notin B.$$

This contradiction completes the proof. $\square$

We now use this to show there is no surjective map $f\colon \mathbb{N} \to [0, 1)$ and therefore $[0, 1)$, and therefore the set of all real numbers, $\mathbb{R}$, is larger than

than the set of natural numbers. That is the set of real numbers is uncountable.

Let $x \in [0,1)$. Then let

$$.x_1x_2x_3x_4\cdots$$

be its decimal expansion. There is a slight problem in that decimal expansions are not unique. For example

$$.2 = .199999999999\cdots$$

This is only a problem when the number x has only a finite number of nonzero digits, in which case if if x_n is the last non-zero digit then we have

$$.x_1x_2\cdots x_n0000000\cdots = .x_1x_2\cdots x_{n-1}(x_n-1)9999999999\cdots.$$

For example

$$.12475900000000\cdots = .12475899999999\cdots$$

To get uniqueness we will choose always choose the finite version, rather than the version ending in an infinite string of 9s. We

Define a function $\varphi\colon [0,1) \to \mathcal{P}(\mathbb{N})$ by

$$\varphi(x) = \{k : x_k < 5 \text{ where } x = x_1x_2x_3x_4x_5\cdots \text{ is the decimal expansion.}\}$$

This function is surjective. For example if $S \subseteq \mathbb{N}$, let

$$x_k := \begin{cases} 3, & k \in S; \\ 6, & k \notin S. \end{cases}$$

and $x = x_1x_2x_3x_4\cdots$. Then

$$\varphi(x) = S.$$

(The map φ is not injective (i.e. one to one).)

Theorem 4. *There is no surjective map* $f\colon \mathbb{N}$ *to* $[0,1)$.

Proof. If $f\colon \mathbb{N} \to [0,1)$ is surjective, then the composition $\varphi\circ f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ would be surjective contradicting Theorem 1. $\square$