Mathematics 554 Homework.

Here is a problem to review some of what we know about metric spaces. The reverse triangle inequality is

$$|d(p,x) - d(q,x)| \leq d(p,q).$$

This implies, as we have seen before,

Proposition 1. *If* $\lim_{n\to\infty} p_n = p$ *in a metric space, then for any* q *we have*

$$\lim_{n\to\infty} d(p_n, q) = d(p,q).$$

Proof. By the reverse triangle inequality

$$|d(p_n,q) - d(p,q)| \leq d(p_n,p).$$

Let $\varepsilon > 0$. Then there is a N so that $n \leq N$ implies $d(p_n, p) < \varepsilon$. But using the reverse triangle inequality just given we have

$$|d(p_n,q) - d(p,q)| \leq d(p_n,p) < \varepsilon$$

which shows $\lim_{n\to\infty} d(p_n,q) = d(p,q)$. □

Remark 2. Anther way to think of this is that the reverse triangle inequality gives that $f(x) := d(x,q)$ is a Lipschitz function with Lipschitz constant $M = 1$. And we have shown that for a Lipschitz function that $\lim_{n\to\infty} p_n = p$ implies $\lim_{n\to\infty} f(p_n) = f(p)$. □

Here is an application of this.

Proposition 3. *Let* A *be a compact subset of the metric space* E *and let* $p \in E$. *Define the distance of* p *from* A *by*

$$\mathrm{Dist}(p,A) = \inf\{d(p,a) : a \in A\}.$$

Show there is an $a_0 \in A$ *with*

$$d(p,a_0) = \mathrm{Dist}(p,A).$$

(That is there is a point of A *closest to* p*.)*

Proof. (This proof was on one of my previous Math 554 exams.) To simplify notation let $D = \mathrm{Dist}(p,A)$. By the definition of $D = \mathrm{Dist}(p,A)$ as an infinitum we have that for each $n \in \mathbb{N}$ there is a point $a_n \in A$ with $D \leq d(p,a_n) < D + 1/n$. This implies $|D - d(p,a_n)| < 1/n$ and thus

$$\lim_{n\to\infty} d(p,a_n) = D.$$

(For it $\varepsilon > 0$ and $N = 1/\varepsilon$, then $n > N$ implies $|D - d(p,a_n)| < \varepsilon$.) As A is compact, it is sequentially compact and therefore there is a subsequence $\langle a_{n_k}\rangle_{k=1}^{\infty}$ so that $\lim_{k\to\infty} d(p,a_{n_k}) = D$ for some $a_0 \in A$. Using that subsequences of convergent sequences converge to the same limit as the original sequence and Proposition 3 we have

$$d(p,a_0) = \lim_{k\to\infty} d(p,a_{n_k}) = D$$

as required. □

Problem 1. Let A be a compact subset of a metric space and let $p \in E$. Define

$$M = \sup\{d(p,a) : a \in A\}.$$

Show that is a point $a_0 \in A$ with

$$d(p, a_0) = M.$$

(That is there is point of A farthest from p.) *Hint:* We have shown that compact sets in metric spaces are bounded and so $M < \infty$. Now use the same ideas as the proof of Proposition 3. $\square$

Proposition 4. *If* $\lim_{n\to\infty} p_n = p$ *and* $\lim_{n\to\infty} q_n = q$ *in a metric space, then*

$$\lim_{n\to\infty} d(p_n, q_n) = d(p,q).$$

Problem 2. Prove this. *Hint:* One one to start to justify the following:

$$\begin{aligned} |d(p_n, q_n) &= |d(p_n,q_n) - d(p,q_n) + d(p,q_n) - d(p,q)| \\ &\le |d(p_n,q_n) - d(p,q_n)| + |d(p,q_n) - d(p,q)| \\ &\le d(p_n, p) + d(q_n, q) \end{aligned}$$

and do an $\varepsilon/2$ proof.

Here are some applications showing one of the many ways compactness is used.

Let S be a set in a metric space (or just in the plane, $\mathbb{R}^2$, if you like). Then the ***diameter*** of S is

$$\text{Diamter}(S) := \sup\{d(p,q) : p, q \in S\}.$$

That is the diameter is the least upper bound on the distances between points in S. We would like to say that it is the greatest possible distance between point of S this is not always the case. For example if $S = (0,1)$, an open interval in $\mathbb{R}$, then $\text{Diameter}(S) = 1$, but if $p, q \in (0,1) = S$ we have $d(p,q) = |p-q| < 1$. On the other hand if is the closed interval $[0,1]$, then the maximum distance is realised, as $0, 1 \in [0,1]$ and $\text{Diameter}([0,1]) = 1 = |1-0|$.

In general we can find points that are at a maximal distance apart in a set when the set is compact.

Proposition 5. *Let S be a compact set in a metric space. Then there are points $p_0, q_0 \in S$ with*

$$\text{Diameter} = d(p_0, q_0).$$

Proof. We have proven that compact sets in metric spaces are bounded, so the diameter is finite. Let $D = \text{Diameter}(S)$. By the definition of Diameter as $\sup(\{d(p,q) : p,q \in S\})$ we have for each $n \in \mathbb{N}$ a pair p_n, q_n such that

$$D - \frac{1}{n} < d(p_n, q_n) \le D.$$

This implies $|D - d(p_n, q_n)| < 1/n$ and therefore

$$\lim_{n\to\infty} d(p_n, q_n) = D.$$

(For, as we have done before, if $\varepsilon > 0$ and $N = 1/\varepsilon$, then $n > N$ implies $|D - d(p_n, q_n)| < 1/n < \varepsilon$.)

$$\lim_{n\to\infty} d(p_n, q_n) = D.$$

As S is compact, it is also sequentially compact. Thus there is a subsequence $\langle p_{n_k}\rangle_{n=1}^{\infty}$ so that

$$\lim_{k\to\infty} p_{n_k} = p_0$$

for some $p \in S$. Now we have to do the subsequence of a subsequence trick. There is a subsequence $\langle q_{n_{k_j}}\rangle_{j=1}^{\infty}$ of $\langle q_{n_k}\rangle_{k=1}^{\infty}$ so that

$$\lim_{j\to\infty} q_{n_{k_j}} = q_0$$

for some q_0 in S. Using that subsequences of convergent sequences converge to the same limit as the ordinal sequence together with Proposition 4 gives

$$d(p_0, q_0) = \lim_{j\to\infty} d(p_{n_{k_j}}, q_{n_{k_j}}) = D.$$

Thus p_0 and q_0 satisfy the conclusion of the proposition. □

Proposition 6. *Let A and B be compact subsets of the metric space and let*

$$M = \inf(\{d(a,b) : a \in A,\ \text{and } b \in B\}).$$

Then there are $a_0 \in A$ and $b_0 \in B$ so that

$$d(a_0, b_0) = M.$$

Problem 3. Prove this. *Hint:* This is very much like the proof of Proposition 5. By the definition of M as an infinitum for each $n \in \mathbb{N}$ there are $a_n \in A$ and $b_n \in B$ such that

$$M \le d(a_n, b_n) < M + \frac{1}{n}$$

and this implies $\lim_{n\to\infty} d(a_n, b_n) = M$. Now use sequential compactness and a subsequence of a subsequence argument to get the existence of a_0 and b_0 so that $d(a_0, b_0) = M$. □

Problem 4. Let A be a subset of $[a,b]$ with $a \in A$ and so that A is both open and closed in $[a,b]$. Show $A = [a,b]$. *Hint:* Toward a contradiction assume that $A \ne [a,b]$. Let $B = [a,b] \setminus A$ be the compliment of A in $[a,b]$. Explain why B is also both open and closed in $[a,b]$. Let $\beta = \inf(B)$ and explain why $\beta \in B$. Then explain why β is an adherent point of A and why this implies $\beta \in A$. We now have that $\beta \in A \cap B$, which contradicts that B is the compliment of A. □

Problem 5. Prove that any compact metric space, E, is complete. *Hint:* You need to show that any Cauchy sequence in E converges. So you probably want to use that compact spaces are sequentially compact. $\square$

The next problem is off of one of my previous Math 554 exams. (It was was during pandemic and thus was take home so time was not an issue.)

Problem 6. Let define a sequence of real numbers $\langle x_n \rangle_{n=1}^{\infty}$ by

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \frac{3}{4}x_1 + 12 \\ x_3 &= \frac{3}{4}x_2 + 12 \\ x_4 &= \frac{3}{4}x_3 + 12 \end{aligned}$$

and in general

$$x_n = \frac{3}{4}x_{n-1} + 12.$$

(a) Compute x_2, x_3, x_4.
(b) Use induction to show the sequence is increasing.
(c) Use induction to show $x_n \le 100$ for all n.
(d) Prove $\langle x_n \rangle_{n=1}^{\infty}$ converges and find its limit. $\square$

In the class where the previous problem was given the Banach Fixed Point Theorem was not covered. Using the fixed point theorem we can generalize the problem.

Problem 7. Recall that a map $f\colon E \to E$ from a metric space to itself is ***contraction*** if and only if there is a constant ρ with $0 \le \rho < 1$ such that

$$d(f(p), f(q)) \le \rho d(p, q)$$

for all $p, q \in E$. The Banach Fixed Point Theorem is that if f is a contraction and the space E is complete, then f has a unique fixed point (that it a point p with $f(p) = p$) and that this fixed point is the limit of an sequence defined by choosing any point $p_0 \in E$ and defining p_n for $n \ge 1$ by

$$x_n = f(x_{n-1}).$$

Then the unique fixed point is the limit

$$p_* = \lim_{n \to \infty} p_n.$$

Now let $a, b \in \mathbb{R}$ be constants with $|a| < 1$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = ax + b.$$

(a) Show f is a contraction.

(b) Therefore the Banach Fixed Point Theorem implies that if x_0 is any real number and for $n \geq 1$

$$x_n = f(x_{n-1}) = ax_{n-1} + b$$

that this sequence converges to the unique fixed point of f. Find the fixed point and in the case where $a = 3/4$ and $b = 12$ compare with the answer to Problem 6. $\square$

Problem 8. (Anther problem off of an old exam.) Let S be the subset of $\mathbb{R}^2$ defined by

$$S = \{(x, y) : 0 < x^2 + y^2 \leq 1\}.$$

(a) Draw a picture of S.
(b) Show that S is not sequentially compact. $\square$

Problem 9. This is anther review problem. Let $f\colon E \to \mathbb{R}$ be Lipschitz, say $|f(p) - f(q)| \leq Md(p, q)$ for all $p, q \in E$. Then for any $c \in \mathbb{R}$ the sets

$$f^{-1}[(-\infty, c)] = \{p : f(p) < c\}, \qquad \text{and} \qquad f^{-1}[(c, \infty)] = \{p : f(p) > c\}$$

are both open. Choose one these two and prove it is open. (No extra points for doing both.) $\square$

Proposition 7. *Let E be a connected metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Assume there are points $p_1, p_2 \in E$ with $f(p_1) < 0$ and $f(p_2) > 0$. Then there is a point $q \in E$ with $f(q) = 0$.*

Problem 10. Prove this. *Hint:* Assume that it is false. Then use Problem 9 to show that

$$A := \{p : f(p) < 0\} \qquad \text{and} \qquad B := \{p : f(p) > 0\}$$

is a disconnection of E, contradicting that E is connected.

Problem 11. Anther problem off of an old exam. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers with $\lim_{n\to\infty} x_n = L$. Give a ε, N proof that $\lim_{n\to\infty} x_n^2 = L^2$.

Problem 12. (a) Define what it means for a subset S of a metric space E to be ***compact***.
(b) Show that if S is compact it can be covered by a finite number of balls of radius .001.

Problem 13. These are to highlight mistakes of "mathematical grammar". Example of such a thing in linear algebra is if c is a scalar and $\mathbf{v}$ is a vector then statements such as

$$c = \mathbf{x}, \qquad c + \mathbf{x}.$$

The first does not make sense as scalars and vectors are just different types of objects and so can not be equal and the second because addition between scalars and vectors is not defined. Here are some examples of bad mathematical grammar that have come up is this class. In each case explain what is wrong with the statement.

(a) If E is a metric space and $p \in B(q,r)$, then $|p-q| < r$.
(b) If $\mathcal{U}$ is an open cover of S and $s \in S$, then $s \in \mathcal{U}$.
(c) If $\mathcal{U}$ is an open cover of S, then $U \in \bigcup \mathcal{U}$.
(d) If $r_1 < r_2$, then $B(p,r_1) \in B(p,r_2)$. $\square$

Problem 14. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space E so that $p = \lim_{n\to\infty} p_n$ exists. Show the set $S = \{p\} \cup \{p_n : n \in \mathbb{N}\}$ is compact. $\square$

Problem 15. Gives examples of

(a) A bounded subset of $\mathbb{R}$ with no maximum.
(b) A sequence in $\mathbb{R}$ with no convergent subsequence.
(c) An open cover of $[0,1)$ with no finite subcover. $\square$

Problem 16. Let E be a disconnected metric space and let

$$E = A \cup B$$

be a disconnection of E. Let S be a connected subset of E. Show that either $S \subseteq A$ or $S \subseteq B$. $\square$