Mathematics 554 Homework.

The problems are 20 points each.

Problem 1. (a) Show that the function

$$f(x) = \frac{x}{1+x^2}$$

is Lipschitz on the closed interval $[-b, b]$ for any $b > 0$.

(b) Use this to give a detailed N, ε proof that if $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} x_n = L$ that

$$\lim_{n\to\infty} \frac{x_n}{x_n^2+1} = \frac{L}{L^2+1}.$$

□

Problem 2. Let (A, d_A) and (B, d_B) be metric spaces. There are many ways to make the Cartesian product $E := A \times B$ into a metric space. One easy way is to use a variant of "taxi geometry" and define a metric d on E by

$$d((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + d_B(b_1, b_2).$$

(a) Prove this is a metric on $E = A \times B$.
(b) Prove that if A and B are complete, then so is E.
(c) Prove that if A and B are both sequentially compact, then so is E. *Hint:* For this you are going to have to use the subsequence of a subsequence trick.

Remark 1. Anther way to make $A \times B$ into a metric space is defining the metric on $E = A \times B$ by the by

$$d((a_1, b_1), (a_2, b_2)) = \sqrt{d_A(a_1, a_2)^2 + d_B(b_1, b_2)^2}.$$

If A and B are both the real numbers with their usual distance $d(x, y) = |x-y|$ then this metric gives the usual Euclidean geometry for the plane.

Proposition 2. *Let (E, d) and (E', d') be metric spaces and let $f \colon E \to E'$ satisfy*

$$d'(f(p), f(q)) \leq M d(p, q)$$

for some constant $M > 0$. (That is f is Lipschitz.) Then, if $V \subseteq E'$ is an open set then

$$U := f^{-1}[V] = \{p \in E : f(p) \in V\}$$

is also open.

Problem 3. Prove this. *Hint:* Let $p \in U$, then $f(p) \in V$.

(a) Use that V is open to explain why there is a $r > 0$ so that $B(f(p), r) \subseteq V$.

(b) Let $\delta = r/M$ explain why the Lipschitz condition implies that if $q \in B(p, \delta)$ then $f(p) \in B(f(p), r) \subseteq V$.
(c) Conclude that $B(p, \delta) \subseteq U$ and explain why this implies U is open. $\square$

Proposition 3. *Let $\mathbb{R}^2$ have its usual metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In different notation if $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ then

$$d(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|.$$

(a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Then the map $f \colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

is Lipschitz.

*(b) For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ define the **segment** with endpoints $\mathbf{a}$ and $\mathbf{b}$ is*

$$[\mathbf{a}, \mathbf{b}] = \{(1 - t)\mathbf{a} + t\mathbf{b} : 0 \leq t \leq 1\}.$$

Then $[\mathbf{a}, \mathbf{b}]$ is connected.

Problem 4. Prove this. *Hint:* If $[\mathbf{a}, \mathbf{b}] = A \cup B$ is a disconnection of $[\mathbf{a}, \mathbf{b}]$ set

$$A_0 = \{t \in [0, 1] : (1-t)\mathbf{A}+t\mathbf{b} \in A\}, \qquad B_0 = \{t \in [0, 1] : (1-t)\mathbf{A}+t\mathbf{b} \in B\}$$

use Part (a) and Proposition 2 to explain why $[0, 1] = A_0 \cup B_0$ is a disconnection of $[0, 1]$ and why this proves the result. $\square$

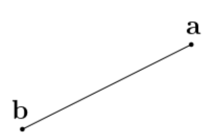

FIGURE 1. The segment between $\mathbf{a}$ and $\mathbf{b}$ is the set of points $(1 - t)\mathbf{a} + t\mathbf{b}$ with $0 \leq t \leq 1$.

Definition 4. Let $\mathcal{F}$ be a collection of subsets of the set E. Then $\mathcal{F}$ has the ***finite intersection property*** if and only if every finite subset $\mathcal{F}_0 = \{F_1, F_2, \dots, F_n\}$ of $\mathcal{F}$ has nonempty intersection. That is

$$\bigcap \mathcal{F}_0 = F_1 \cap F_2 \cap \cdots \cap F_n \neq \varnothing.$$

The last problem is really just an advanced Math 300 problem in that it uses the definitions and some logic.

Proposition 5. *Let E be a metric space.*

(a) Show that if E is compact, then any any collection, $\mathcal{F}$, of closed subsets of E with the finite intersection property has nonempty intersection. That is if $\mathcal{F}$ has the finite intersection property, then

$$\bigcap \mathcal{F} \neq \varnothing.$$

(b) Conversely, if every collection of closed sets, $\mathcal{F}$, of E with the finite intersection property has nonempty intersection, then E is compact.

Problem 5. Prove this. *Hint for (a):* Assume this is false, that is $\mathcal{F}$ has the finite intersection property, but $\bigcap \mathcal{F} = \varnothing$. Let $\mathcal{U} = \{\mathcal{C}(F) : F \in \mathcal{F}\}$. Show $\bigcap \mathcal{F} = \varnothing$ implies that $\mathcal{U}$ is an open cover of E. Take a finite sub cover $\{\mathcal{C}(F_1), \mathcal{C}(F_2), \ldots, \mathcal{C}(F_n)\}$ and show $F_1 \cap F_2 \cap \cdots \cap F_n = \varnothing$, contradicting that $\mathcal{F}$ has the finite intersection property. *Hint for (b):* Let $\mathcal{U}$ be an open cover of E and assume that $\mathcal{U}$ does not have a finite sub cover. Let $\mathcal{F} := \{\mathcal{C}(U) : U \in \mathcal{U}\}$. Let why $\mathcal{U}$ not having any finite sub cover implies that $\mathcal{F}$ is a collection of closed sets with the finite intersection property. But then $\bigcap \mathcal{F} \neq \varnothing$. Explain why this contradicts that $\mathcal{U}$ is an open cover of E. □

Remark 6. The statement "a collection of closed sets with the finite intersection property has nonempty intersection" is really nothing more than the contrapositive of the De Morganization of the statement "an open cover has a finite sub cover".