Mathematics 554 Homework.

Read Sections 4.1–4.3 of *Notes on Analysis* pages 77–82.

Problem 1. Give an ε, δ proof that $f(x) = x^3 - x$ is continuous at all points a. $\square$

Problem 2. Give a ε, δ proof that the function $f(x) = \sqrt{|x|}$ continuous at $x = 0$. $\square$

Problem 3. Give an ε, δ proof that $f(x) = \dfrac{x}{1+x}$ is continuous at any point $a \neq -1$. $\square$

Problem 4. Let $g \colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that g is not continuous at any point. $\square$

Problem 5. Do Problem 4.5 on Page 79 of the text.

Problem 6. (a) Show that the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ x \cos(1/x), & x > 0. \end{cases}$$

is continuous at 0. *Hint:* You are allowed to assume that $|\cos(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$.

(b) Show that the function

$$g(x) = \begin{cases} 0, & x \leq 0 \\ \cos(1/x), & x > 0. \end{cases}$$

is not continuous at $x = 0$. *Hint:* You are allowed to assume that $\cos(2\pi n) = 1$ for all $n \in \mathbb{Z}$.

Proposition 1. *Let (E, d) and (E', d') be metric spaces and assume that f is continuous at the point p_0. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in E with $\lim_{n \to \infty} p_n = p_0$. Then*

$$\lim_{n \to \infty} f(p_n) = f(p_0).$$

Problem 7. Prove this. $\square$