Mathematics 554 Homework.

We have just proven the following:

Theorem 1. *Let E and E' be metric spaces and $f\colon E \to E'$ a function and $p_0 \in E$. Then the following are equivalent.*

(a) *f is continuous at p_0.*
(b) $\lim_{p\to p_0} f(p) = f(p_0)$
(c) *f does right by sequences converging to p_0. That is if $\langle p_n\rangle_{n=1}^{\infty}$ is a seqnence in E with $\lim_{n\to\infty} p_n = p_0$, then $\lim_{n\to\infty} f(p_n) = f(p_0)$.*
(d) *If V is an open set in V and $f(p_0) \in V$, then $f^{-1}[V]$ contains an open ball about p_0. That is there is an $r > 0$ so that $B_E(p_0, r) \subseteq f^{-1}[V]$.* □

As we have only just defined it here is the official definition of limit $\lim_{p\to p_0} f(p) = q_0$.

Definition 2. Let E, E' be metric spaces and $f\colon E \to E'$ a function. Let $p_0 \in E$ and $q_0 \in E'$. Then

$$\lim_{p\to p_0} = q_0$$

if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ so that for $q \in E'$,

$$0 < d(p, p_0) < \delta \qquad \text{implies} \qquad d'(f(p), q_0) < \varepsilon,$$

□

Definition 3. Let E and E' be metric spaces and $f\colon E \to E'$ a function. Then f is ***continuous*** if and only if f is continuous at every point of E. □

Theorem 4 (Continuous functions are great)**.** *Let E and E' be metric spaces and $f\colon E \to E'$ a function. Then the following are equivalent.*

(a) *f is continuous.*
(b) *for every $p_0 \in E$ the limit $\lim_{p\to p_0} f(p) = f(p_0)$ holds.*
(c) *f does right by all convergent sequences in E, That is if $\langle p_n\rangle_{n=1}^{\infty}$ is a sequence in E with $\lim_{n\to\infty} p_n = p_0$, then $\lim_{n\to\infty} f(p_n) = f(p_0)$.*
(d) *If V is an open subset of E', then the preimage $f^{-1}[V]$ is an open subset of E.*

Problem 1. Prove this. *Hint:* That

$$(a) \iff (b) \iff (c)$$

follows at once from the equivalence of (a), (b), and (c) in Theorem 1. So you only have to prove (a) $\iff$ (d). Do this in detail. □

Problem 2. Let $f\colon E \to E'$ be a continuous function between metric spaces and $\mathcal{U}$ an open cover of E'. Prove $\{f^{-1}[V] : V \in \mathcal{U}\}$ is an open cover of E. □

You have probably seen false proofs such as

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (i)(i) = -1$$

and

$$1 = \frac{d}{dx}x = \frac{d}{dx}\underbrace{(1+1+\cdots+1)}_{x\text{ terms}}. = \underbrace{0+0+\cdots+0}_{x\text{ terms}} = 0$$

Here is a variant of false proofs I have seen in homework.

Problem 3. What is wrong with with the following proof that

$$\lim_{x\to 1}\frac{1}{x-1} = 0.$$

Let $\varepsilon > 0$. Set $\delta = |x-1|^2\varepsilon$. If $0 < |x-1| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x-1} - 0\right| &= \frac{1}{|x-1|}\\ &= \frac{1}{|x-1|^2}|x-1|\\ &< \frac{0}{|x-1|^2}\delta && \text{(as } |x-1| < \delta\text{)}\\ &= \frac{1}{|x-1|^2}|x-1|^2\varepsilon\\ &= \varepsilon.\end{aligned}$$

This if $f(x) = 1/(x-1)$ the inequality $0 < |x-1| < \delta$ implies $|f(x) - 0| < \varepsilon$ which verifies that the definition of $\lim_{x\to 1} = 0$ holds. □

Problem 4. Show that

$$\lim_{x\to 1}\frac{1}{x-1}$$

does not exist. □