Review for Test 3.

For practice doing the problems on Test 3 on
`http://ralphhoward.github.io/Classes/Spring2021/554/`
and the problems on Test 3 of
`http://ralphhoward.github.io/Classes/Fall2018/554/`
The second of these pages also has a review for its Test 3 that has some good worked problems.

Problem 1. Definitions you should know:

(a) The metric space E is ***connected***. (This includes knowing what a ***disconnection*** of a metric is.
(b) The definition of a metric space being ***compact***.
(c) The definition of a metric space being ***sequentially compact***.
(d) The definition of $\lim_{n\to\infty} p_n = p$.
(e) The definition of $\lim_{p\to p_0} f(p) = q_0$ where $f\colon E \to E'$ is a map between metric spaces.
(f) If $f\colon E \to E'$ is a map between metric spaces know the definition of f is ***continuous at*** p_0,
(g) Likewise that say $f\colon E \to E'$ being ***continuous*** means that f is continuous at all points of E.
(h) Know what it means for $\mathcal{U}$ to be an ***open cover*** of a set S.
(i) It would not hurt to review what is means for a set to be open or closed and what it means for p to be an adherent point of a set S.

Here are some theorems you should know.

Theorem 1. *A subset of a metric space is compact if and only if it is sequentially compact.* □

Theorem 2 (Heine-Borel Theorem)**.** *A closed bounded subset of $\mathbb{R}^n$ is compact.* □

Theorem 3. *An interval in $\mathbb{R}$ is connected.* □

Theorem 4 (Intermediate Value Theorem)**.** *If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a)$ and $f(b)$ have opposite signs, then there is a point $\xi \in [a,b]$ with $f(\xi) = 0$.* □

Theorem 5. *The continuous image of a connected set is connected. That is if $S \subseteq E$ is connected and $f\colon E \to E'$ is a continuous map between metric spaces, then $f[S]$ is a connected subset of E'.* □

Theorem 6. *The continuous image of a compact set is compact. That is if $f\colon E \to E'$ is a continuous map between metric space and $K \subseteq E$ is compact, then $f[K]$ is a compact subset of E'.* □

Theorem 7 (Existence of extreme values)**.** *If E is a compact metric space and $f\colon E \to \mathbb{R}$ is continuous, then f has a maximum and minimum. That* *is there are* $p_0, p_1 \in E$ *such that*

$$f(p_0) \leq f(p) \leq f(p_1)$$

for all $p \in E$. *Thus* $f(p_0)$ *is the minimum value of* f *and* $f(p_1)$ *is the maximum value of* f *on* E.