Mathematics 554 Homework.

The following should be a review from from Math 300 related material.

Proposition 1. *Let $f\colon A \to B$ be a bijection between sets. (Recall that "bijection" is the same as "one to one onto"). Then f has an inverse. That is there is a function $g\colon B \to A$ such that for all $a \in A$ and all $b \in B$*

$$f(g(b)) = b, \qquad \text{and} \qquad g(f(a)) = a.$$

*This function is unique and is the **inverse** of f. It is written as $g = f^{-1}$.*

Problem 1. Prove this. □

Problem 2. Show that if $f\colon A \to B$ is a bijection, then the inverse $f^{-1}\colon B \to A$ is also a bijection and $(f^{-1})^{-1} = f$. □

The following is a useful fact about the inverses of continuous functions on compact spaces.

Proposition 2. *Let $f\colon E \to E'$ be a continuous bijection between metric spaces with E compact. Then the inverse $f^{-1}\colon E' \to E$ is continuous.*

Problem 3. Prove this. *Hint:* The continuous image of a compact set is compact and f is surjective thus E' is compact. One of our equivalent condition for a function being continuous is that the preimage of closed sets is closed. Let $K \subseteq E'$ be closed, then it is enough to show $f[K] = (f^{-1})^{-1}[K]$ is a closed subset of E. □

Example 3. Here is a example to show that Proposition 2 can fail when E is not compact. Let $E = [0, 2\pi)$ and $E' = \{(x, y) : x^2 + y^2 = 1\}$ and define f by $f(t) = (\cos(t), \sin(t))$. This is bijective, continuous, but the inverse is not continuous at $t = 0$. □

Problem 4. Let $f\colon [a, b] \to [\alpha, \beta]$ be an increasing (that is $x_1 < x_2$ implies $f(x_1) < f(x_2)$ continuous function with $f(a) = \alpha$ and $f(b) = \beta$. Prove that f is bijective and that the inverse f^{-1} is continuous. □

Proposition 4. *If $K_1, K_1, K_3 \subseteq \mathbb{R}^2$ are compact subsets of $\mathbb{R}^2$, then*

$$K = K_1 \times K_2 \times K_3$$

is a compact subset of $\mathbb{R}^6$

Proof. By the Heine-Borel Theorem it is enough to show K is closed and bounded. Each of K_1, K_2, K_3 is compact and therefore they are closed and bounded. From this it is not hard to see that K is bounded. To show it is closed it is enough to show it contains the limits of all its convergent sequences. So let

$$\langle \mathbf{p}_k \rangle_{k=1}^{\infty} = \langle (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) \rangle_{k=1}^{\infty}$$

be a sequence of points from K with

$$\lim_{k\to\infty} \mathbf{p}_k = \lim_{k\to\infty} (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) = (\mathbf{x}, \mathbf{y}, \mathbf{z})$$

where $\mathbf{x} = (x_{1k}, x_{2k})$, $\mathbf{y} = (y_{1k}, y_{2k})$, and $\mathbf{z} = (z_{1k}, z_{2k})$. Our goal is to prove $\mathbf{p} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in K$. Then, variants of arguments we have done before, show that the above limit implies the three

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}, \quad \lim_{k\to\infty} \mathbf{y}_k = \mathbf{y}, \quad \lim_{k\to\infty} \mathbf{z}_k = \mathbf{z}.$$

As K_1, K_2, and K_3 are close they contain their limit points and thus $\mathbf{x} \in K_1$, $\mathbf{y} \in K_2$, $\mathbf{z} \in K_3$ and so $\mathbf{p} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in K_1 \times K_2 \times K_3 = K$ and were are done. $\square$

If $x = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{z} = (z_1, z_2)$. The the area of the triangle with vertices $\mathbf{z}$, $\mathbf{y}$, and $\mathbf{z}$ is

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{2}\|(\mathbf{x} - \mathbf{z}) \times (\mathbf{y} - \mathbf{z})\|$$

where $\times$ is the cross product

$$(a, b, c) \times (x, y, z) = (bz - cy, -az + cx, as - bx).$$

This is a continuous function as it only involves multiplication, addition, and subtraction. Then $A(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a continuous function of as it only involves some extra multiplications and taking a square root and the composition of continuous functions is continuous.

Problem 5. Let K be a closed bounded subset of $\mathbb{R}^2$. Show that there exists $\mathbf{x}_8, \mathbf{y}_*, \mathbf{z}_* \in K$ so that the triangle $\triangle \mathbf{x}_* \mathbf{y}_* \mathbf{z}_*$ has maximum area of all triangles with vertices in K. That is

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq A(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*)$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in K$. $\square$