Mathematics 554 Homework.

Here are what I consider the high lights of the course. I will take at least half of the final off of these questions. You should know a proof of any result that where the proof is given here. You do no have to give the same proof I as the one here, but you should know how to do at least one proof.

To start here is a list of things you should certainly know.

(a) The sum of a geometric series and how to factor $x^n - y^n$.
(b) The definition of a metric space.
(c) The ε–N definition of the limit of a sequence.
(d) The ε–δ definition of a limit for functions between metric spaces.
(e) The ε–δ proof of a function being continuous at a point.
(f) The definition of a function being Lipschitz.
(g) You will have to do at least one of either a ε–N or ε–δ proof.

Problem 1. (a) Define what it means for b to be the ***supremum*** (that is ***least upper bound***) of $S \subseteq \mathbb{R}$.
(b) State the least upper bound axiom.
(c) Define what it means for p to be an ***adherent point*** of the set S.
(d) Prove that a closed set contains all its adherent points.
(e) If $S \subseteq \mathbb{R}$ is bounded above prove that $\sup(S)$ is an adherent point of S.
(f) If $S \subseteq \mathbb{R}$ define what it means for b to be the ***maximum*** of S.
(g) Prove that a closed bounded subset of $\mathbb{R}$ has a maximum.
(h) Give an example of a bounded subset of $\mathbb{R}$ that does not have a maximum.

Solution. (a) b is an upper bound for S (that is $s \leq b$ for all $s \in S$) and if c is any upper bound for S, then $b \leq c$. □

(b) Any nonempty subset of $\mathbb{R}$ which is bounded above, has a supremum. □

(c) If S is a subset of a metric space E, then p is an adherent point of S if and only if for each $r > 0$ we have $B(p,r) \cap S \neq \varnothing$. (Here $B(p,r) = \{x \in E : d(p,x) < r\}$ is the open ball of radius r about p.) □

(d) Let S be a closed subset of the metric space E. Towards a contradiction assume that there is an adherent point, p, of S which is not in S. Then p is in the compliment, $\mathcal{C}(S)$, of S and as S is closed the compliment is open. By definition of open set this means there is an $r > 0$ so that $B(p,r) \subseteq \mathcal{C}(S)$. But then $B(p,r) \cap S = \varnothing$, contradicting that p is an adherent point of S. □

(e) Let $S \subseteq \mathbb{R}$ be bounded above and let $b = \sup(S)$. Let $r > 0$. Then $B(b,r) = (b-r, b+r)$. Choose any $x \in (b-r, b)$. Then $x < b$ and b is the least upper bound of S and therefore x is not an upper bound for S. Thus there is $s \in S$ with $x < s < b$. Then $s \in B(b,r) \cap S$ and so $B(p,r) \cap S \neq \varnothing$. This works for all $r > 0$ and so b is an adherent point of S. □

(f) b the maximum of S if and only if $b \in S$ and $s \leq b$ for all $b \in S$. □

(g) Let S be a closed bounded subset of $\mathbb{R}$. As S is bounded $b = \sup(S)$ exists. We have just seen that b is an adherent point of S. As S is closed we have also just seen that this implies $b \in S$. Then for $s \in S$ we have $s \neq b$ as b is an upper bound for S (in fact the least upper bound). Thus b is the maximum of S. □

(h) A natural example is the open interval $S := (0, 1)$. Then $\sup(S) = 1$, but $1 \notin S$ so S has not maximum. □

Problem 2. (a) Let E be a metric space and $K \subseteq E$. Define $\mathcal{U}$ is an ***open cover*** of K.

(b) Show that for any $r > 0$ that

$$\mathcal{U} := \{B(p, r) : p \in K\}$$

is an open cover of K. (On the final I would let you by with just saying this is the true, but make sure that you understand why. If you are having trouble with this, you will have trouble with other questions.)

(c) Show that for any point $p_0 \in E$ that

$$\mathcal{U} := \{B(p_0, r) : r > 0\}$$

is an open cover of E (and thus also of any subset of E). (This is anther fact you could use on the final without proof, but again make sure you know how to prove it.)

(d) Let $f : E \to \mathbb{R}$ be a continuous function. Show that

$$\mathcal{U} := \{f^{-1}[(-r, r)] : r \in \mathbb{R}\}$$

is an open cover of E and thus any subset of E.

Solution. (a) $\mathcal{U}$ is an open cover of K if and only if each $U \in \mathcal{U}$ is an open subset of E and for all $p \in K$ there is a $U \in \mathcal{U}$ with $p \in U$. (Or what is the same thing, each element of $\mathcal{U}$ is open and $K \subseteq \bigcup \mathcal{U}$.) □

(b) Each element, $B(p, r)$, is an open ball and therefore an open set. And if $p \in K$, then $p \in B(p, r) \in \mathcal{U}$. Thus $\mathcal{U}$ is an open cover. □

(c) Each element, $B(p_0, r)$, is an open ball and thus open. Let $p \in E$ and let choose $r > d(p, p_0)$. Then $p \in B(p_0, r)$ and $B(p_0, r) \in \mathcal{U}$. □

(d) As f is continuous and $(-r, r)$ is open in $\mathbb{R}$ the set set $f^{-1}[(-r, r)]$ is open as the preimages of open sets by continuous functions is open. Thus each element of $\mathcal{U}$ is open. Let $p \in E$. The choose $r \in \mathbb{R}$ with $r > |f(p)|$, say $r = |f(p)| + 1$. Then $p \in f^{-1}[(-r, r)]$ and $f^{-1}[(-r, r)] \in \mathcal{U}$. Thus $\mathcal{U}$ is an open cover of E. □

Problem 3. This is not a problem, just a list of a few things we have proven that you can use in proofs without having to reprove.

(a) Let $r_1 r_2, \ldots, r_n$ be positive real numbers, set

$$r_{\max} = \max\{r_1, r_2, \ldots, r_n\}, \qquad r_{\min} = \min\{r_1, r_2, \ldots, r_n\}$$

E a metric space and $p \in E$. Then

$$\bigcup_{j=1}^{n} B(p, r_j) = B(p, r_{\max}), \qquad \bigcap_{j=1}^{\infty} B(p, r_j) = B(p, r_{\min}).$$

and if $f\colon E \to \mathbb{R}$ is a function, then

$$\bigcup_{j=1}^{\infty} f^{-1}[(-\infty, r_j)] = f^{-1}[(-\infty, r_{\max})]$$
$$\bigcap_{j=1}^{\infty} f^{-1}[(-\infty, r_j)] = f^{-1}[(-\infty, r_{\min})]$$

(b) That taking preimages does the right thing to set theoretic operations. That is

$$f^{-1}\left[\bigcup_{\alpha \in A} U_\alpha\right] = \bigcup_{\alpha \in A} f^{-1}\left[U_\alpha\right]$$
$$f^{-1}\left[\bigcap_{\alpha \in A} U_\alpha\right] = \bigcap_{\alpha \in A} f^{-1}\left[U_\alpha\right]$$
$$f^{-1}[\mathcal{C}(U)] = \mathcal{C}(f^{-1}[U])$$

This includes things like if $A \cap B = \varnothing$, then $f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = f^{-1}[\varnothing] = \varnothing$.

(c) A sum of squares is positive. The is $x^2 + xy + y^2 = (x + y/2)^2 + (3/4)y^2$ is positive.

Let us now put the information above to good use.

Proposition 1. *If K is a compact subset of a metric space E, then K is closed in E. (Recall a subset of a metric space is **bounded** if and only if it is contained in some ball.)*

Proof. We can choose any point p_0 of E to be the center of our ball. Then

$$\mathcal{U} = \{B(p_0, r) : r > 0\}$$

is an open cover of all of E, and therefore a open cover of K. It therefore has a finite subcover

$$\mathcal{U}_0 = \{B(p_0, r_1), B(p_0, r_2, \ldots, B(p_n, r_n)\}$$

that covers K. That is

$$K \subseteq \bigcup_{j=1}^{n} B(p_0, r_j) = B(p_0, r_{\max})$$

where $r_{\max} = \max\{r_1, r_2, \ldots, r_n\}$. Thus K is bounded. □

Proposition 2. *If K is a compact subset of the metric space E, then K is closed in E.*

Proof. Towards a contraction assume that K is not closed. Then K has an adherent point p that is not in K. For each $r > 0$ let

$$U_r = \{q \in E : d(p,q) > r\} = \mathcal{C}(\overline{B}(p,r)).$$

That is U_r is the set of points at a distance $> r$ from p, which is just the compliment of the closed ball $\overline{B}(p,r)$ and thus is an open set. Let

$$\mathcal{U} := \{U_r : r > 0\}$$

then each U_r in $\mathcal{U}$ is open, and of $q \in K$, then $q \neq p$ (as $p \notin K$). Choose $r < d(p,q)$, then $q \in U_r$. Thus U_r is a open over of K. It therefore has a finite subcover

$$\mathcal{U}_0 = \{U_{r_1}, U_{t_2}, \ldots, U_{r_n}\}.$$

As $r < s$ implies $U_r \supseteq U_s$ (larger r gives smaller sets U_r) and therefore

$$K \subseteq \bigcup_{j=1}^{n} U_{r_j} = U_{r_{\min}}$$

where $r_{\min} = \min\{r_1, r_2, \ldots, r_n\}$. But then form the definition of $U_{r_{\min}}$ as the compliment of the closed ball $\overline{B}(p, r_{\min})$ we have $K \subseteq U_{r_{\min}}$ implies $B(p, r_{\min}) \cap K = \varnothing$ contradicting that p is an adherent point of K. □

Problem 4. Given an example of a metric space E and a closed bounded subset K of E that is not compact.

Solution. Let $E = \mathbb{Q}$, the rational numbers. Then $K = \{x \in \mathbb{Q} : 0 \leq x \leq 2\} = \mathbb{Q} \cap [0,2]$ closed an bounded in E. To see this is not compact for $r > 0$ let

$$U_r = \{x \in \mathbb{Q} : |x - \sqrt{2}| > r\}$$

then U_r is open in $\mathbb{Q}$ and if $x \in \mathbb{Q}$, then $x \neq \sqrt{2}$ as $\sqrt{2}$ is irrational. So there is r with $0 < r < |x - \sqrt{2}|$ and therefore $x \in U_r$. Thus

$$\mathcal{U} = \{U_r : r > 0\}$$

is an open over of all of $\mathbb{Q}$ and therefore of K. If

$$\mathcal{U}_0 = \{U_{r_1}, U_{r_2}, \ldots, U_{r_n}\}$$

a finite subset of $\mathcal{U}$ then

$$\bigcup_{j=1}^{n} U_{r_j} = U_{r_{\min}}.$$

But there is a rational number in $[0,2]$ between $\sqrt{2}$ and $\sqrt{2} + r_{\min}$, there fore this finite subset of $\mathcal{U}$ does not cover K. Therefore K is not compact. □

But we do have some spaces where the class of closed bounded subsets is the same as the compact subsets. One of the class' most important results is

Theorem 3 (Heine–Borel Theorem)**.** *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.* □

Definition 4. A metric space, E, is ***sequentially compact*** if and only if every $\langle p_n \rangle_{n=1}^{\infty}$ in E has a subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ that converges to point in E.

Anther important result is

Theorem 5. *A matrix space is compact if and only if it is sequentially compact.* □

The proof of this was not as elementary as many of our results. Showing that sequentially compact metric space used the Lebesgue Covering Lemma and may be about the trickiest proof we did. On the other hand showing that compact implies sequentially compact is anther good application of using a good open cover to use to get an even better finite subcover.

Proposition 6. *If the metric space E is compact, then it is sequentially compact.*

Proof. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in the metric space in the compact space E.

Claim. There is a point $p_0 \in E$ so that the set

$$N(p_0, r) = \{n \in \mathbb{N} : p_n \in B(p_0, r)\}$$

is infinite for any $r > 0$. (A little more informally $N(p, r)$ is just the set of n so that the points p_n are a distance less that r from p. Thus the claim is that there is point p_0 so that for any $r > 0$ the ball $B(p_0, r)$ contains infinitely many points of the sequence.)

Towards a contraction, assume that this is false. Then for each $p \in E$ there an $r_p > 0$ so that $N(p, r_p)$ is finite. Then the more or less "obvious" open cover to use is

$$\mathcal{U} = \{B(p, r_p) : p \in E\}.$$

Each element of $\mathcal{U}$ is an open ball and thus open and if $p \in E$, then $p \in B(p, r_p) \in \mathcal{U}$. So $\mathcal{U}$ is an open cover. Then there is a finite subset

$$\mathcal{U}_0 = \{B(p_1, r_1), B(p_1, r_2), \ldots, B(p_n, r_m)\}$$

(where $r_j = r_{r_j}$) that covers E. Then, by how the balls were chosen, each $N(p_j, r_j)$ is finite for $j = 1, 2, \ldots, m$. But for all $n \in \mathbb{N}$ $p_n \in B(p_j, r_j)$ for at least one j (this is because $\mathcal{U}_0$ covers E). Thus

$$\mathbb{N} = N(p_1, r_1) \cup N(p_2, r_2) \cup \cdots \cup N(p_n, r_n).$$

We have therefore have the infinite set $\mathbb{N}$ as a finite union of finite sets which is a contradiction.

The rest of the proof is easy. Let p_0 be as in the claim. Then for any $r > 0$ there are infinitely many p_n with $d(p_0, p_n) < r$. So choose n_1, with $d(p_{n_1}, p_0) < 1$. Then choose $n_2 > n_1$ with $d(p_{n_2}, p_0) < 1/2$ and continuing in this manner we choose $n_k > n_{k-1}$ and with $d(p_{n_k}, p_0) < 1/k$. Then if $\varepsilon > 0$ and $N > 1/\varepsilon$ then $k > N$ implies $1/k < \varepsilon$ and therefore

$$k > N \qquad \text{implies} \qquad d(p_{n_k}, p_0) < \varepsilon.$$

Thus $\lim_{k\to\infty} p_{n_k} = p_0$. Thus we have started with an arbitrary sequence in E an shown it has a convergent subsequence. So E is sequentially compact. □

Definition 7. A metric space, E, is ***complete*** if and only if every Cauchy sequence in E converges.

Make sure you know the definition of a Cauchy sequence and a standard question is to show that a convergent sequence is Cauchy.

Problem 5. (a) Using the least upper bound axiom to show that a bounded monotone sequence in $\mathbb{R}$ is convergent.

(b) We have shown that every sequence in $\mathbb{R}$ has a monotone subsequence. Use this to show that closed bounded subset of $\mathbb{R}$ is sequentially compact. (That is prove the $n = 1$ special case of the Heine–Borel Theorem.)

(c) We have shown that a Cauchy sequence is bounded and that if it has a convergent subsequence it is convergent. (It would not hard to know how to prove this.) Use this to that $\mathbb{R}$ is complete.

Solution. (a) Let $\langle x_n\rangle_{n=1}^{\infty}$ be a bounded monotone sequence in $\mathbb{R}$. We assume that the sequence is monotone increasing. (Otherwise replace the sequence by $\langle -x_n\rangle_{n=1}^{\infty}$). As the sequence is bounded the least upper bound

$$b = \sup\{x_n : n \in \mathbb{N}\}$$

exists. Let $\varepsilon > 0$. Then $b - \varepsilon$ is less than the least upper bound of $\{x_n : n \in \mathbb{N}\}$ and therefore is not an upper bound for the set. Thus there is a $N \in \mathbb{N}$ so that $b - \varepsilon < x_N \le b$. If $n \ge N$ then the sequence being monotone increasing implies $b - \varepsilon < p_N \le p_n \le b$ which implies $|p_n - b| < \varepsilon$. Thus $n \ge N$ implies $|p_n - b| < \varepsilon$ and so $\lim_{n\to\infty} x_n = b$. □

(b) If $\langle x_n\rangle_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, then it has a monotone subsequence, $\langle x_{n_k}\rangle_{k=1}^{\infty}$. We have just seen that such a sequence has is convergent, so we are done. □

(c) Let $\langle x_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. We wish to show it converges. As it is Cauchy it is bounded. But we have seen that a bounded sequence has a convergent subsequence. But a Cauchy sequence with a convergent subsequence is convergent. □

One of our main theorems about continuous functions is:

Theorem 8. *Let $f\colon E \to E'$ be a function. Then the following are equivalent:*

(a) f is continuous. (By which we mean that f is continuous at every point of E.)

(b) f does the right thing to convergent sequences in E. (That is if $\lim_{n\to\infty} p_n = p$ in E, then $\lim_{n\to\infty} f(p_n) = f(p)$ in E'.

(c) f does the right thing to limits. (That is $\lim_{p\to p_0} f(p) = f(p_0)$.)

(d) *The preimages of open sets by f are open. (That is if $V \subseteq E'$ is open, then so is the set $f^{-1}[V] = \{p \in E : f(p) \in V\}$).*
(e) *The preimages of closed sets by f are closed. (That is if $F \subseteq E'$ is closed, then so is the set $f^{-1}[F] = \{p \in E : f(p) \in F\}$.)* □

Problem 6. (a) Show the set $U = \{(x, y) : 2x^2 - 3y^4 > 0\}$ is an open set in $\mathbb{R}^2$.
(b) Show the set $C = \{(x, y, z) : x^4 + y^4 + z^4 = 1$ is compact.

Solution. (a) The function $f(x, y) = 2x^2 - 3y^2$ is continuous as it is a polynomial. Then $U = f^{-1}[(0, \infty)]$ and $(0, \infty)$ is open in $\mathbb{R}$. Use U is the preimage of an open set by a continuous function and therefore U is open. □
(b) The function $f(x, y, z) = x^4 + y^4 + z^4$ is a polynomial and therefore continuous. Then $C = f^{-1}[\{1\}]$ is the preimage of the closed set $\{0\}$ and is therefore closed. Note that on C $x^4 = 1 - y^4 - z^4 \leq 1$. And $x^4 \leq 1$ implies $|x| \leq 1$. Likewise $|y| \leq 1$ and $|z| \leq 1$. Therefore C is bounded. Thus C is closed and bound and therefore compact by the Heine-Borel Theorem. □
□

Theorem 9. *If E is compact and $f : E \to E'$ is continuous, then the image $f[E]$ is compact.*

Solution. Let $\mathcal{V}$ be an open over of $f[E]$ and let

$$\mathcal{U} = \{f^{-1}[V] : V \in \mathcal{V}\}.$$

Then each $f^{-1}[V]$ is the preimage of an open set by a continuous function and therefore the elements of $\mathcal{U}$ are open. If $p \in E$, then $f(p) \in f[E]$ and $\mathcal{V}$ covers $f[E]$ so $f(p) \in V$ for some $V \in \mathcal{V}$. Then $p \in f^{-1}[V] \in \mathcal{U}$. Thus $\mathcal{U}$ is an open cover of E. Thus, as E is compact, there is a finite subset

$$\mathcal{U}_0 = \{f^{-1}[V_1], f^{-1}[V_2], \dots, f^{-1}[V_n]\}$$

of $\mathcal{U}$ covers E. We now show that

$$\mathcal{V}_0 = \{V_1, V_2, \dots, V_n\}$$

is cover of $f[E]$. Let $q \in f[E]$. Then $q = f(p)$ for some $p \in E$. Because $\mathcal{U}_0$ covers E there is a j so that $p \in f^{-1}[V_j]$. Then $q = f(p) \in \mathcal{V}_0$. Thus shows $\mathcal{V}_0$ covers $f[E]$. We started with an arbitrary open cover $\mathcal{V}$ of $f[E]$ and showed it has a finite subset $\mathcal{V}_0$ which covers $f[E]$. Therefore $f[E]$ is compact. □

Definition 10. Let E be a metric space. The E is ***connected*** if and only if E is not the disjoint union of two nonempty open set. □

Definition 11. Let E be a metric space. Then a ***disconnection*** of E is a pair of sets $A, B \subseteq E$ so that

- A and B are open,
- $A \neq \varnothing$ and $B \neq \varnothing$,

- $A \cup B = E$, and
- $A \cap B = \varnothing$. □

Thus a restatement of E being connected is that it does not have a disconnection.

Theorem 12. *If EE is a connected metric space, and $f\colon E \to E'$ is continuous, then the image $f[E]$ is connected.*

Proof. By replacing E' with the image $f[E']$ we can assume that f is surjective. Towards a contradiction assume that $f[E]$ has a disconnection $f[E] = A \cup B$ where A and B are disjoint nonempty open subsets of E'. Then, as taking perimages preserves set theoretic operations,

$$E = f^{-1}[E'] = f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$$

and

$$f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = f^{-1}[\varnothing] = \varnothing.$$

Each of $f^{-1}[A]$ and $f^{-1}[B]$ are open sets as they are preimages of open sets by a continuous function. Finally $A \neq \varnothing$ so that is an $a \in A$. As f is surjective there is a $p \in E$ with $f(p) = a$. Then $p \in f^{-1}[A]$ and so $f^{-1}[A] \neq \varnothing$. Likewise $f^{-1}[B] \neq \varnothing$. Thus $E = f^{-1}[A] \cup f^{-1}[B]$ is a disconnection of E, contradicting that E is connected. □

We were able to find all the connected subsets of $\mathbb{R}$.

Theorem 13. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.* □

After just a little be of work proving this was reduced to proving:

Lemma 14. *Let $[a,b]$ be a closed interval in $\mathbb{R}$. Then is no disconnection $[a,b] = A \cup B$ of $[a,b]$ with $a \in A$ and $b \in B$.*

Proof. Towards a contradiction assume there is such a disconnection. First note that A is open in $[a,b]$ by the definition of disconnection. Also A is closed in $[a,b]$ as it is the compliment of the open set B. As A is bounded above (by b) it has a least upper bound. Let

$$c = \sup(A).$$

Then, as A is closed, we have that $c \in A$. But A is also open so there is a ball $B(c,r) = (c-r, c+r) \subseteq A$ Then the points in the subinterval $(c, c+r) \subseteq A$. But a point $x \in (c, c+r)$ has $x \in A$ and $x > c$ contradicting that c is the least upper bound for A. This contradiction completes the proof. □

Finally, let us give a general form of the intermediate value theorem.

Theorem 15. *Let E be a connected metric space and $f\colon E \to \mathbb{R}$ a continuous function. Let $p_0, p_1 \in E$ be points with with $f(p_0) < f(p_1)$. Then for any $c \in \mathbb{R}$ with $f(p_0) < c < f(p_1)$ there is a $p \in E$ with $f(p) = c$.*

Proof. As f is continuous the image $f[E]$ is a connected subset of $\mathbb{R}$. But we have seen the connected subsets of $\mathbb{R}$ are the intervals. A basic property of intervals is if two points are in the interval, then so are all the points between them. This implies that $f[E]$ contains all the points between $f(p_0)$ and $f(p_1)$ and thus the point c. But if c is in the image there is a point $p \in E$ with $f(p) = c$. □

A second proof. Towards a contradiction assume there is no $p \in E$ with $f(p) = c$. Let

$$A = (-\infty, c), \qquad B = (c, \infty)$$

These are disjoint open subsets of $\mathbb{R}$ and $A \cup B = \mathbb{R} \setminus \{c\}$. As f dose not take on the value c we have $f[E] \subseteq A \cup B$. The sets $f^{-1}[A]$ and $f^{-1}[B]$ are open as they are the preimages of open sets by a continuous function. Also $p_0 \in f^{-1}[A]$ and $p_1 \in f^{-1}[B]$. Thus both $f^{-1}[A]$ and $f^{-1}[B]$ are nonempty. Finally $f[E] \subseteq A \cup B$ implies

$$E = f^{-1}[A] \cup f^{-1}[B].$$

Therefore $E = f^{-1}[A] \cup f^{-1}[B]$ is a disconnection of E, contradicting that E is connected. □