Mathematics 554 Homework.

Problem 1. Find the sums of the following series:

(a) $\sum k = 1^{20} x^3 (1-x^2)^k$.

(b) $\sum_{j=0}^{n} (3j+1)$.

(c) $\sum_{k=0}^{n} 2^k \binom{n}{k}$.

Problem 2. (a) State the binomial theorem.

(b) Simplify $\dfrac{(a+h)^4-(a-h)^4}{2h}$. (The answer should not have an h in the denominator.)

Problem 3. (a) Define what it means for a function $f\colon [a,b] \to \mathbb{R}$ to be ***Lipschitz***.

(b) Show that the function $f(x) = 3x^2-1$ is Lipschitz on the interval $[-4,3]$.

(c) Show that $g(x) = \dfrac{x+3}{x+4}$ is Lipschitz on $[1,5]$.

(d) Show the function $h(x) = \sqrt{x}$ is not Lipschitz on $[0,1]$.

Problem 4. Show $x^2 - 2xy + 2y^2 \geq 0$ with equality if and only if $x = y = 0$. *Hint:* There are many ways to do this, but one natural one involves completing the square.

Problem 5. (a) State the Intermediate Value Theorem for Lipschitz functions on an interval $[a,b]$.

(b) Show the $p(x) = x^4 + 3x - 1$ has at least two real roots. (You may assume polynomials are Lipschitz.)

Problem 6. Show that if $|x| \geq \max\{1, 2(|a|+|b|)\}$, then

$$1 + \frac{a}{x} + \frac{b}{x^2} \geq \frac{1}{2}.$$

Problem 7. (a) State the Least Upper Bound Axiom.

(b) Give an example of a set which is bounded both above and below, that has a maximum but no minimum.

(c) Let $S \subset [0,\infty)$ be bounded above and let

$$T = \{s^2 : s \in S\}.$$

Show

$$\sup(T) = (\sup(S))^2.$$

Hint: One way is to prove the two inequalities $\sup(T) \leq (\sup(S))^2$ and $(\sup(S))^2 \leq \sup(T)$. The first of these follows without too much trouble from definitions. For $(\sup(S))^2 \leq \sup(T)$ let $\varepsilon > 0$ and show $(\sup(S)-\varepsilon)^2 \leq \sup(T)$ holds.

Problem 8. (a) State the big version of Archimedes Axiom.
(b) Prove Archimedes from the Least Upper Bound axiom.

Problem 9. Let $S \subset \mathbb{R}$ have the property that if $s \in S$, there is anther $s' \in S$ with $s' > s + 1/2$. Show S is not bounded above.

Problem 10. Solve the inequality $\dfrac{3x-1}{1-2x} \geq 4$.

Problem 11. (a) State the Cauchy-Schwartz inequality for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.
(b) Use the Cauchy-Schwartz inequality to prove the Triangle Inequality $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$.

Problem 12. (a) Define (E, d) is a ***metric space***.
(b) Let $p \in E$ and $r > 0$ define $B(p, r)$ (the ***open ball*** of radius r about p), and the $\overline{B}(p, r)$ (the ***closed ball*** of radius r about p).