Mathematics 554 Homework.

Definition 1. A metric space (E, d) is ***complete*** if and only if every Cauchy sequence in E converges to a point of E. □

We have seen that the real numbers, $\mathbb{R}$, is a complete metric space with its usual metric $d(x, y) = |x - y|$.

If (E, d) is a metric space and $F \subseteq E$ is a subset of E, the (F, d) is also a metric space. (Really this should be written as $(F, d\big|_{F\times F})$ where $d\big|_{F\times F}$ is the restriction of d from $E \times E$ to $F \times F$, but I do not think the extra notation makes things any clearer.)

Proposition 2. *If (E, d) is a metric space and F is a closed subset of E, then (F, d) is also a complete metric space.*

Problem 1. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in F. Then this sequence is also a Cauchy sequence in E. As E is complete, we have $\lim_{n\to\infty} p_n = p$ for some $p \in E$. Review what we have proven about closed sets to find a result that lets us conclude that $p \in F$. □

Problem 2. (See Problem 3.50 in the notes for hints on this.) Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^3$ with its usual metric. Let $p_n = (x_n, y_n, z_n)$.

(a) Show that each of the sequence $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$, and $\langle z_n \rangle_{n=1}^{\infty}$ are also Cauchy sequences and explain why this implies the limits $x := \lim_{n\to\infty} x_n$, $y := \lim_{n\to\infty}$, and $z := \lim_{n\to\infty} z_n$ exist.

(b) Let $p = (x, y, z)$ and show

$$\lim_{n\to\infty} p_n = p.$$

(c) Conclude that $\mathbb{R}^3$ is a complete metric space. □

Problem 3. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E. Prove that the sequence is bounded. That is show there is a ball $B(p, r)$ (or if you like a closed ball $\overline{B}(p, r)$) with $p_n \in B(p, r)$ (or $p_n \in \overline{B}(p, r)$) for all n. □

Definition 3. Let E be a metric space and $f \colon E \to E$ a function form E to itself. Then f is a ***contraction*** if and only if there is a real number ρ with $0 \leq \rho < 1$ so that

$$d(p, q) \leq \rho d(p, q)$$

for all $p, q \in E$.

Put somewhat differentially a function $f \colon E \to E$ is a contraction if and only if it is a Lipschitz map with Lipschitz constant less than 1.

Theorem 4 (Banach Fixed Point Theorem)**.** *Let E be a complete metric space and $f \colon E \to E$ a contraction. Then there is a unique point $p_* \in E$ with*

$$f(p_*) = p_*.$$

More concisely this says a contraction on a complete metric space has a unique fixed point. (By definition a ***fixed point*** of a function, f, is a point p with $f(p) = p$.)

Problem 4. This is a special case of a result we have done before, but a little review is not a bad thing. Let $f\colon E \to E$ be a contraction and let $\lim_{n\to\infty} p_n = p$ in E. Show

$$\lim_{n\to\infty} f(p_n) = f(p).$$

□

Problem 5. Prove the Banach Fixed Point Theorem along the following lines. (If you do not want to work with metric spaces, it is fine with me if you do the special case of f a contraction $f\colon \mathbb{R} \to \mathbb{R}$.)

To start choose any point $p_0 \in E$ and define a sequence $\langle p_n \rangle_{n=1}^{\infty}$ by

$$p_1 = f(p_0), \quad p_2 = f(p_1), \quad p_3 = f(p_2), \ldots \quad p_{n+1} = f(p_n), \ldots$$

(a) Show for $k \geq 1$ that

$$d(p_k, p_{k+1}) \leq \rho^k d(p_0, p_1).$$

(b) If $m < n$ show

$$d(p_m, p_n) \leq \frac{\rho^m - \rho^n}{1 - \rho} d(p_0, p_1) \leq \frac{\rho^m}{1 - \rho} d(p_0, p_1)$$

Hint: Start by recalling that when we first introduced metric spaces that we showed the triangle inequality implies

$$d(p_m, p_n) \leq \sum_{k=m}^{n-1} d(p_k, p_{k+1}).$$

(So you do not have to prove this again.) Now use the bound of part (a) and that you know how to sum a finite geometric series.

(c) Show if N is a natural number and $m, n \geq N$, then

$$d(p_m, p_n) \leq \frac{\rho^N}{1 - \rho} d(p_0, p_1)$$

(d) We have shown that if $0 \leq \rho < 1$, then $\lim_{N\to\infty} \rho^N = 0$ back when we first talked about the least upper bound axiom. It follows that

$$\lim_{N\to\infty} \frac{\rho^N}{1 - \rho} d(p_0, p_1) = 0.$$

(You do not have to prove this again.) Use this to show $\langle p_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

(e) As E is complete this implies $\langle p_n \rangle_{n=1}^{\infty}$ is convergent. Let $p_* := \lim_{n\to\infty} p_n$. Use $p_{n+1} = f(p_n)$ and Problem 4 to show $f(p_*) = p_*$. This shows the existance of a fixed point for f.

(f) Show the fixed point is unique. *Hint:* If p_* and p_{**} are fixed points of f, then $d(p_*, p_{**}) = d(f(p_*), f(p_{**}))$ and use that f is a contraction to show $d(p_*, p_{**}) = 0$. □

Note that the proof of the Banach Fixed Theorem not only shows the existence of the fixed point, it gives a method of finding it as a limit by starting with any $p_0 \in E$ and forming the sequence $p_{n+1} = f(p_n)$. Then $p_* = \lim_{n\to\infty} p_n$ is the fixed point.

Problem 6. Let $a \geq 1$ and define $f\colon [0,\infty) \to [0,\infty)$ by

$$f(x) = \sqrt{a+x}.$$

(a) Show for $x, y \in [0,\infty)$ that

$$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{a+x}+\sqrt{a+y}} \leq \frac{|x-y|}{2\sqrt{a}} \leq \frac{1}{2}|x-y|$$

and therefore f is a contraction. The space $[0,\infty)$ is a complete metric space as it is a closed subset of the complete space $\mathbb{R}$ (See Proposition 2 above.)

(b) Define a sequence $x_0 = a$ and $x_{n+1} = f(x_n)$. Then

$$x_0 = \sqrt{a}$$
$$x_1 = \sqrt{a+\sqrt{a}}$$
$$x_2 = \sqrt{a+\sqrt{a+\sqrt{a}}}$$
$$x_3 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$$
$$x_4 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}}$$
$$x_5 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}}}$$
$$x_6 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}}}}$$
$$x_7 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}}}}}$$

(Yes, I did get a little carried away, but I had a macro do most of the work and I had fun writing the macro). The Banach Fixed Point Theorem tells us this converges to the unique fixed point of f. Find the this fixed point. Note this limit can be interpreted as giving meaning to

$$\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}.$$

Problem 7. The Banach Fixed Point Theorem can be used to solve equations that at first glace are not fixed point problems. As an example let us compute numerically a root of the equation

$$x^3 - 5x - 1 = 0.$$

We can rewrite this as

$$\frac{x^3 - 1}{5} = x$$

so we are looking for a fixed point of f given by

$$f(x) = \frac{x^3 - 1}{5}.$$

Let $E = [-1, 1]$. This is a closed subspace of $\mathbb{R}$ and therefore is a complete metric space.

(a) If $|x| \leq 1$ show

$$|f(x)| \leq \frac{2}{5}$$

and therefore f maps E into E.

(b) Show if $x, y \in E$ (that is $|x|, |y| \leq 1$) then

$$|f(x) - f(y)| \leq \frac{3}{5}|x - y|$$

and therefore f is a contraction on $E = [-1, 1]$.

(c) We can now approximate the fixed point of f, which will be the unique root of $x^3 - 5x - 1$ in $[-1, 1]$, by letting $x_0 = 0$, and $x_{n+1} = f(x_n)$. Doing 10 steps in this gives

n	x_n
0	0.0
1	-0.2
2	-0.2016
3	-0.2016387080192
4	-0.20163965211624296
5	-0.20163967514750483
6	-0.20163967570935554
7	-0.20163967572306193
8	-0.2016396757233963
9	-0.20163967572340447
10	-0.20163967572340463

This already gives over 10 decimal places of accuracy. (Generally the convergence is not this fast, but with a computer doing several hundred, or even several thousand, steps is not a problem.)