Mathematics 554H/703I Test 2 Name: Answer Key
Show your work to get credit.

1. (10 points) State the following:
(a) The ***Bolzano Weierstrass Theorem***.

Solution. I accepted several different versions:

- (Preferred version) A closed bounded subset of $\mathbb{R}^n$ is compact. (or is sequentially compact.)
- A closed bounded subset of $\mathbb{R}$ is compact. (or is sequentially compact.)
- A bounded sequence in $\mathbb{R}^n$ (or of $\mathbb{R}$) has a convergent subsequence. $\square$

(b) The definition of $\mathcal{U}$ is an ***open cover*** of S.

Solution. Is S is a subset of the metric space E then $\mathcal{U}$ is an open cover of S if and only if

(a) Each $U \in \mathcal{U}$ is a open subset of E,
(b) $S \subseteq \bigcup \mathcal{U}$ (or equivalently for each $s \in S$ there is a $U \in \mathcal{U}$ with $s \in U$). $\square$

(c) S is a ***bounded*** subset of a metric space.

Solution. S is a bounded subset of E if and only if there is $p \in E$ and $r > 0$ so that $S \subseteq B(p, r)$. (Equivalently: S is bounded if and only if it is contained in a ball.) *Note:* Saying there are numbers a and b so that $a \leq s \leq b$ for all $s \in S$ does not make sense in a general metric space. That is only a good definition of bounded in $\mathbb{R}$. $\square$

2. (10 points) Use the least upper bound axiom to give a N, ε proof that a bounded monotone increasing sequence is convergent.

Solution. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a monotone increasing sequence in $\mathbb{R}$ which is bounded. Let $b = \sup\{x_n : n \in \mathbb{N}\}$, which exists as the sequence is bounded. We now show $\lim_{n\to\infty} x_n = b$ which will show the sequence converges. Let $\varepsilon > 0$. Then these is a $N \in \mathbb{N}$ so that $b - \varepsilon < p_N \leq b$ (otherwise $b - \varepsilon$ would be an upper bound of $\{x_n : n \in \mathbb{N}\}$ less than the least upper bound b). As the sequence is monotone increasing if $n \geq N$, then $b - \varepsilon < x_N \leq x_n \leq b$. This implies $|x_n - b| < \varepsilon$. Therefore $n \geq N$ implies $|x_n - b| < \varepsilon$. That is $\lim_{n\to\infty} x_n = b$. $\square$

3. (15 points) (a) Let E be a metric space. Define S is a ***compact*** subset of E.

Solution. The set is a compact subset of E if and only if every open cover of S has a finite subcover. $\square$

(b) Let S be a compact subset of E and assume S is infinite. Prove there is a point $p \in S$ so that $S \cap B(p, r)$ is infinite for all $r > 0$.

Solution using open covers. Towards a contradiction assume that for each $p \in S$ there is an $r_p > 0$ so that $S \cap B(p, r_p)$ is finite. Then (as we have seen before)

$$\mathcal{U} := \{B(p, r_p) : p \in S\}$$

is an open cover of S. As S is compact there is a finite subcover

$$\mathcal{U}_0 = \{B(p_1, r_1), B(p_2, r_2), \ldots, B(p_n, r_n)\}$$

where we have simplified notation by setting $r_j := r_{p_j}$. As $S \subseteq \bigcup \mathcal{U}_0$ we have

$$S = S \cap \bigcup_{j=1}^{n} B(p_j, r_j) = \bigcup_{j=1}^{n} (S \cap B(p_j, r_j)).$$

Thus S is is finite union of finite sets, contradicting that S is infinite. $\square$

Solution using sequential compactness. As S is compact it is sequentially compact. As S is infinite we can choose a sequence $\langle p_n \rangle_{n=1}^{\infty}$ where $p_i \neq p_j$ when $i \neq j$. (To see this start by choosing any $p_1 \in S$, then, as S is infinite, there is a $p_2 \in S$ with $p_2 \neq p_1$, and we can keep going, having chosen $p_1, p_2, \ldots, p_n \in S$ all distinct from each other, there is a $p_{n+1} \in S \setminus \{p_1, p_2, \ldots, p_n\}$ as S is infinite.) As S is sequentially compact there is a convergent subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ with $\lim_{k \to \infty} p_{n_k} = p$ for some $p \in S$. Let $r > 0$, then (by letting $\varepsilon = r$ in the definition of limit) there is a N so that if $k > N$, then $p_{n_k} \in B(p, r)$. Then $B(p, r)$ contains the infinite set $\{p_{n_k} : k > N\}$ and as all of the points p_{n_k} are in S we see that $B(p, r) \cap S$ is infinite for all $r > 0$. $\square$

4. (15 points) Let E be a compact metric space and $f \colon E \to \mathbb{R}$ a function with the property that if $\langle p_n \rangle_{n=1}^{\infty}$ is a convergent sequence in E, then

$$\lim_{n \to \infty} f(p_n) = f\big(\lim_{n \to \infty} p_n\big).$$

Let $M = \sup_{p \in E} f(p)$. Show there is a point $p_0 \in E$ with $f(p_0) = M$.

Solution. As E is compact, it is sequentially compact. We will find the point p_0 as the limit of a subsequence of a sequence of cleverly chosen points in E. Here is the clever choice:[1] for each $n \in \mathbb{N}$ there is a point p_n so that

$$M - \frac{1}{n} < f(p_n) \le M.$$

This follows from the definition of the supremum (otherwise $M - 1/n$ would be an upper bound less that the least upper bound). Note $|f(p_n) - M| < 1/n$ and so by a now familiar argument

$$\lim_{n\to\infty} f(p_n) = M.$$

As the space is sequentially compact there is subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$ which converges to a point of E, say

$$\lim_{k\to\infty} p_{n_k} = p_0.$$

This defines the point, p_0, we are looking for. We now verify it does what we want:

$$f(p_0) = f\Big(\lim_{k\to\infty} p_{n_k}\Big) = \lim_{k\to\infty} f(p_{n_k}) = M$$

and we are done. □

5. (20 points) (a) Define what it means for A, B to be a ***disconnection*** of the metric space E.

Solution. The set A and B a disconnection of E if and only if

(a) A and B are nonempty,
(b) A and B are both open,
(c) $A \cup B = E$, and
(d) $A \cap B = \varnothing$.

□

(b) Define what it means for a metric space E to be ***connected***.

Solution. The space E is connected if and only if it has no disconnection. □

(c) Recall that a set A in a metric space is ***clopen*** if and only if it is both open and closed. Prove a metric space is connected if and only if the only clopen sets in E are E and $\varnothing$.

Solution. We prove the equivalent statement that a metric space is disconnected if and only if it contains a clopen set other than E and $\varnothing$.

[1]This choice is not really very clever once you have seen several problems of this type.

First assume that E has a disconnection $E = A \cup B$. Then A is open by the definition of disconnection and also form the definition of disconnection we have that $A = E \setminus B$ is the compliment of B in E. As B is open this implies A is closed. Thus A is clopen. Finally $A \neq \varnothing$ by the definition of disconnection and $A \neq E$ as $B \neq \varnothing$ and $A \cap B = \varnothing$ we have $A \neq E$. So E contains a clopen set other than E and $\varnothing$.

Second assume that E contains a clopen set A other that E and $\varnothing$. Let $B = E \setminus A$ be the compliment of A in E. As A is closed and B is the compliment of A, we have that B is open. As $A \neq E$ we have that $B \neq \varnothing$. As B is the compliment of A it follows $A \cap B = \varnothing$ and $E = A \cup B$. Therefore A, B is a disconnection of E. □

6. (10 points) Show the only nonempty connected subsets of the rational numbers, $\mathbb{Q}$, are the one point sets $\{p\}$. *Hint:* You may assume that between any two rational numbers, there is an irrational number.

Solution. If a set has a disconnection $A \cup B$ then as each of A and B is nonempty the set $A \cup B$ has at least two points. Therefore a one point set has no disconnection and is thus connected. So all the one point subsets of $\mathbb{Q}$ are connected.

So we only need to show that any subset of $\mathbb{Q}$ which has more than one point is disconnected. Let $E \subseteq \mathbb{Q}$ have at least two points. Let $a, b \in E$ with $a \neq b$. We can assume that $a < b$. Then there is an irrational number c with $a < c < b$. Let

$$A = (-\infty, r) \cap E = \{x \in E : x < c\}$$
$$B = E \cap (r, \infty) = \{x \in E : c < x\}.$$

Then by their constitution and that $c \notin E$ we have

$$E = A \cup B, \qquad A \cap B = \varnothing.$$

Also $A, B \neq \varnothing$ as $a \in A$ and $b \in B$.

As this point I was ok with someone just saying that it is clear that A and B are open. If you want a blow by blow here is is. To see A is open note if $x \in A$ and $r = (c - x)$, then the ball $B(x, r)$ in A is

$$\begin{aligned} B(x,r)i &= \{y \in E : |x - y| < r = (c - x)\} \\ &= E \cap (x - r, x + r) \\ &= E \cap (2x - c, c) \subseteq A. \end{aligned}$$

Thus A contains a ball around each of its points and so is open. A similar argument shows B is open. Therefore $E = A \cup B$ is a disconnection of E and so E is not connected. □

7. (10 points) (a) Let S be a set in the metric space E. Define what if means for p to be an ***adherent point*** of S.

Solution. The point p is an adherent point of S if and only if for all $r > 0$

$$S \cap B(p, r) \neq \varnothing. \qquad \square$$

(b) Prove that if S is closed and p is an adherent point of S, then $p \in S$.

Solution. Towards a contradiction assume that $p \notin S$. As S is closed the compliment $\mathcal{C}(S)$ is open. As $p \in \mathcal{C}(S)$ and (S) is open there is a ball $B(p, r) \subseteq \mathcal{C}(S)$. But then $B(p, r) \cap S = \varnothing$, which contradicts that p is an adherent point of S. $\square$

8. (10 points) Give examples of the following (no proofs required):

(a) An open cover of $\mathbb{R}$ with no finite subcover.

Solution. Here are several examples

$$\{(-r, r) : r > 0\}, \qquad \{(-n, n) : n \in \mathbb{N}\}, \qquad \{(n, n+2) : n \in \mathbb{Z}\}.$$

In all three of these example, if $\mathcal{U}_0$ is a finite subset of the cover, the union, $\bigcup U_0$ is a bounded interval and thus not all of $\mathbb{R}$. $\square$

(b) A subset of $\mathbb{R}$ that is neither open or closed.

Solution. We have done many examples of this. Here are some $[0, 1)$, $\mathbb{Q}$, $\{1/n : n \in \mathbb{N}\}$. $\square$

(c) A subset S of $\mathbb{R}$ and a point p so that p is an adherent point of S by $p \notin S$.

Solution. One example is $S = (0, 1]$ and $p = 0$. Anther is $S = \mathbb{Q}$ and p any irrational number. $\square$